

# Emission from the D1D5 CFT

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**ABSTRACT:** It is believed that the D1D5 brane system is described by an ‘orbifold CFT’ at a special point in moduli space. We first develop a general formulation relating amplitudes in a  $d$ -dimensional CFT to absorption/emission of quanta from flat infinity. We then construct the D1D5 vertex operators for minimally coupled scalars in supergravity, and use these to compute the CFT amplitude for emission from a state carrying a single excitation. Using spectral flow we relate this process to one where we have emission from a highly excited initial state. In each case the radiation rate is found to agree with the radiation found in the gravity dual.

**KEYWORDS:** Black Holes in String Theory, AdS–CFT Correspondence, Conformal Field Models in String Theory.

## 1. Introduction

The D1D5 system has been very useful in the study of black holes [1, 2, 3, 4, 5, 6]. This system arises from a bound state of D1 branes and D5 branes. The near horizon geometry of these branes is dual to a 1+1 dimensional CFT. While the 3+1 dimensional  $\mathcal{N} = 4$  Yang–Mills CFT (arising from D3 branes) has been extensively studied [7, 8, 9], there has been much less work with the 1+1 dimensional CFT [10].

The goal of this paper is to set up a formalism for computing amplitudes in the orbifold CFT and relating them to absorption/emission of quanta from the gravitational solution describing the D-branes. This requires two main steps. For the first, note that the CFT describes only the physics in the ‘near-horizon region’ of the branes; vertex operators in the CFT create excitations that must travel through the ‘neck’ of the D-brane geometry and then escape to infinity as traveling waves. Thus we set up a general formalism that relates CFT amplitudes to absorption/emission rates observed from infinity<sup>1</sup>. For the second step, note that early computations of radiation [1, 2, 3, 4, 5, 6] used the somewhat heuristic picture of an ‘effective string’ to describe the D1D5 bound state. We construct states and vertex operators in the orbifold CFT, setting up notation and tools that allow us to compute amplitudes with ease. We apply these steps to compute the emission rate of supergravity scalars from particular D1D5 states. In particular we can compute emissions in cases where it was unclear how to proceed with the effective string model. The CFT amplitudes, converted to radiation rates by our general formalism, show exact agreement with the emission rates in the dual gravitational geometry.

Specifically, we perform the following steps:

- (i) Traditionally, one uses AdS/CFT to compute correlation functions in the CFT and compares them to quantities computed in the AdS geometry, but we are interested in finding the interactions of the brane system with quanta coming in from or leaving to *flat* infinity. Thus we must consider the full metric of the branes, where at large  $r$  the *AdS* region changes to a ‘neck’ and finally to flat space. We write a general expression for  $\Gamma$ , the rate of radiation to infinity, in terms of the CFT amplitude for the decay process.
- (ii) As an example of our CFT techniques we consider minimal scalars in the D1D5 geometry. An example of such scalars is given by the graviton with indices along the compact torus directions. We construct the correctly normalized vertex operators for these scalars, which are obtained by starting with a twist operator in the CFT and dressing it with appropriate modes of the chiral algebra.
- (iii) We use the notion of ‘spectral flow’ to map states from the Ramond sector (which describes the D1D5 bound state) to the NS sector (which has the vacuum  $|\emptyset\rangle_{NS}$ ). This

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<sup>1</sup>In [11], the connection between waves coming from infinity and operators in the CFT was established for  $l = 0$ ; however, the effect of the ‘neck’ was ignored since it is irrelevant for minimal scalars.

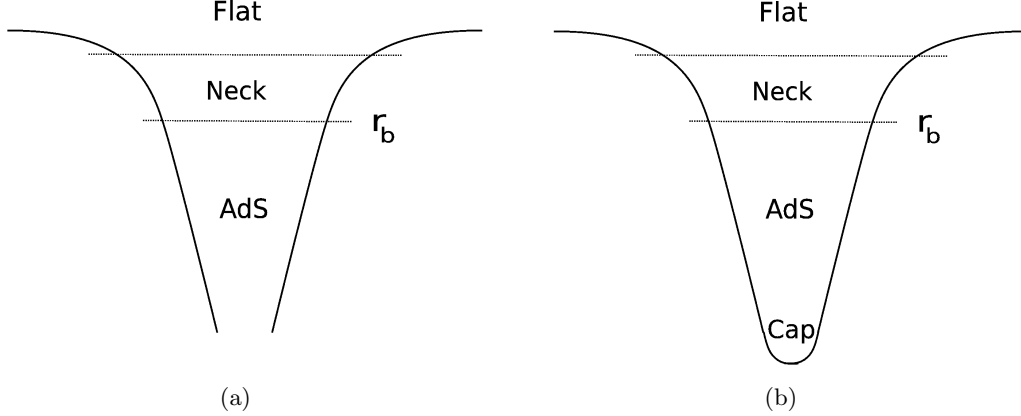
map helps us in two ways. First, it is not clear which states in the Ramond sector correspond to supergravity excitations (as opposed to string excitations). In the NS sector there *is* a simple map: the supergravity excitations are given by chiral primaries and their descendants under the anomaly-free subalgebra of the chiral algebra. Thus the spectral flow map allows us to find the initial and final states of our emission process, given the quantum numbers of these states. Second, for the process of interest the final state turns out to be the *vacuum* in the NS description; thus we do not need an explicit vertex operator insertion to create this state when computing amplitudes after spectral flowing to the NS sector.

- (iv) We consider a simple decay process where an excited state of the CFT decays to the ground state and emits a supergravity quantum. We compute the CFT amplitude for this emission process. As mentioned above, the final state in the amplitude is nontrivial in the Ramond sector, but maps to the NS vacuum  $|\emptyset\rangle_{NS}$  under spectral flow. This converts a 3-point correlator in the Ramond sector to a 2-point correlator in the NS sector. With this amplitude, we use the result in (i) to compute  $\Gamma$ , the rate of emission to flat space.
- (v) In the above computations we take the initial state to contain a few excitations above the ground state, and we compute the decay rate for these excitations. Alternatively, we can choose to start with the initial state having *no* excitations in the NS sector, and then perform a spectral flow on this state. This spectral flow adds a large amount of energy to the state, giving a configuration which is described in the dual gravity by a *nonextremal* classical metric [12]. This metric is known to emit radiation by ergoregion emission [13], and in [14, 15, 16] it was shown that for a subset of possible supergravity emissions the CFT rate agreed exactly with the gravity emission rate. We can now extend this agreement to all allowed emissions of supergravity quanta by using the orbifold CFT computations developed in the present paper. It turns out that the simple decay process computed in steps (iii) and (iv) can be used to give the emission rate from this highly excited initial state. This is because using spectral flow the initial state can be mapped to the vacuum, and in fact the entire amplitude maps under spectral flow to a time reversed version of the decay amplitude computed above. We find exact agreement with the radiation rate in the dual gravity description.

## 2. Coupling flat space fields to the CFT

Consider the geometry traditionally written down for branes that have a near-horizon  $AdS_{d+1} \times S^p$  region. This geometry has the form

$$ds^2 = H^{-\frac{2}{d}} \left[ -dt^2 + \sum_{i=1}^{d-1} dy_i dy_i \right] + H^{\frac{2}{p-1}} [dr^2 + r^2 d\Omega_p^2]. \quad (2.1)$$



**Figure 1:** (a) The geometry of branes is flat at infinity, then we have a ‘neck’, and further in the geometry takes the form  $AdS_{d+1} \times S^p$  (b) Still further in, the geometry ends in a ‘fuzzball cap’ whose structure is determined by the choice of microstate. For the simple state that we will choose for the D1D5 system, the Cap+AdS region is just a part of global  $AdS$ .

If there is only one kind of brane producing the metric (and hence only one length scale) the function  $H$  is given by

$$H = 1 + \frac{Q}{r^{p-1}}. \quad (2.2)$$

The BPS black holes studied in string theory are constructed from  $\mathcal{B}$  sets of mutually BPS branes. In these cases  $H$  is given by

$$H = \prod_{i=1}^{\mathcal{B}} \left( 1 + \frac{Q_i}{r^{p-1}} \right)^{\frac{1}{\mathcal{B}}}. \quad (2.3)$$

(This reduces to (2.2) for  $\mathcal{B} = 1$ .) Let  $Q_{\max}$  be the largest of the  $Q_i$  and  $Q_{\min}$  the smallest. It is convenient to define the length scale

$$R_s = \left( \prod_{i=1}^{\mathcal{B}} Q_i \right)^{\frac{1}{\mathcal{B}(p-1)}}. \quad (2.4)$$

For small  $r$  the angular directions give a sphere with radius  $R_s$ .

We picture such a geometry in Fig. 1(a). The geometry has three regions:

(i) *The outer region:* For large  $r$ ,

$$r \gg Q_{\max}^{\frac{1}{p-1}}, \quad (2.5)$$

we have essentially flat space.

(ii) *The intermediate region:* For smaller  $r$  we find a ‘neck’, which we write as

$$CQ_{\min}^{\frac{1}{p-1}} < r < DQ_{\max}^{\frac{1}{p-1}}; \quad (C \ll 1, \quad D \gg 1). \quad (2.6)$$

(iii) *The inner region:* For

$$r < CQ_{\min}^{\frac{1}{p-1}}, \quad (2.7)$$

we can replace the harmonic function by a power:

$$H \rightarrow \left(\frac{R_s}{r}\right)^{p-1}. \quad (2.8)$$

The directions  $y_i$  join up with  $r$  to make an  $AdS$  space, and the angular directions become a sphere of constant radius:

$$ds^2 \approx \left[ \left(\frac{r}{R_s}\right)^{\frac{2(p-1)}{d}} \left(-dt^2 + \sum_{i=1}^{d-1} dy_i dy_i\right) + R_s^2 \frac{dr^2}{r^2} \right] + R_s^2 d\Omega_p^2, \quad (2.9)$$

which is  $AdS_{d+1} \times S^p$ . To see this we define the new radial coordinate  $\tilde{r}$  by

$$\left(\frac{\tilde{r}}{R_s}\right) = \left(\frac{r}{R_s}\right)^{\frac{p-1}{d}}. \quad (2.10)$$

In terms of this new radial coordinate one has the inner region metric,

$$ds^2 \approx \left[ \left(\frac{\tilde{r}}{R_s}\right)^2 \left(-dt^2 + \sum_{i=1}^{d-1} dy_i dy_i\right) + \left(R_s \frac{d}{p-1}\right)^2 \frac{d\tilde{r}^2}{\tilde{r}^2} \right] + R_s^2 d\Omega_p^2. \quad (2.11)$$

We can scale  $t, y_i$  to put the  $AdS$  into standard form, but it is more convenient to leave it as above, since the coordinates  $t, y_i$  are natural coordinates at infinity and we need to relate the  $AdS$  physics to physics at infinity. The radius of  $AdS_{d+1}$  and the sphere  $S^p$  are given by

$$R_{AdS_{d+1}} = R_s \frac{d}{p-1}, \quad R_{S^p} = R_s. \quad (2.12)$$

In region (iii), we can put a boundary at

$$\tilde{r} = \tilde{r}_b, \quad (2.13)$$

which we regard as the boundary of the  $AdS$  space. We can then replace the space at  $\tilde{r} < \tilde{r}_b$  by a dual CFT. Thus, traditional AdS/CFT calculations are carried out only with the region  $\tilde{r} < \tilde{r}_b$ . Our interest, however, is in the emission and absorption of quanta between the  $AdS$  region and asymptotic infinity. Thus we need a formalism to couple quanta in region (i) to the CFT.

Next, we note that this ‘traditional’  $AdS$  geometry cannot be completely right. In the case of the D1D5 system, we know that the ground state has a large degeneracy  $\sim \exp[2\sqrt{2}\pi\sqrt{N_1 N_5}]$ . At sufficiently large  $r$ , all these states have the form (2.1). But at smaller  $r$  these states differ from each other. None of the states has a horizon; instead each ends in a different ‘fuzzball cap’ [12, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31,

32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43, 44, 45]. We can choose special states where this cap is given by a classical geometry. In the simplest case the cap is such that the entire region  $\tilde{r} < \tilde{r}_b$  has the geometry of *global*  $AdS_3 \times S^3$ . We picture the full geometry for this state in Fig. 1(b).

The geometry without the cap (Fig. 1(a)) would have needed to be supplemented with a boundary condition for the gravity fields at  $\tilde{r} = 0$ . But since the actual states of the system (e.g. Fig. 1(b)) have ‘caps’, we do have a well defined duality between the physics of the region  $\tilde{r} < \tilde{r}_b$  and the CFT at  $\tilde{r}_b$ .

## 2.1 The coupling between gravity fields and CFT operators

Let us start with region (iii), where we have the  $AdS/CFT$  duality map. This map says that the partition function of the CFT, computed with sources  $\phi_b$ , equals the partition function for gravity in the AdS, with field values at the boundary equal to  $\phi_b$  [8]:

$$\int D[X] e^{-S_{\text{CFT}}[X] + \mu \int d^d y \sqrt{g_d} \phi_b(y) \mathcal{V}(y)} = \int D[\phi] e^{-S_{\text{AdS}}[\phi]} \Big|_{\phi(\tilde{r}_b) = \phi_b}. \quad (2.14)$$

Here we have rotated to Euclidean spacetime, setting  $t = y_d$ . The symbol  $\phi$  denotes fields in gravity, and  $\hat{\mathcal{V}}$  are CFT operators (depending on  $X$ ) coupling to the gravity fields. Our first task is to determine the coupling constant  $\mu$ , for the normalizations of  $\phi_b$  and  $\hat{\mathcal{V}}$  that we choose.

Let us discuss the angular dependence in more detail. On  $S^p$  the fields  $\phi(\tilde{r}, y, \Omega)$  can be decomposed into spherical harmonics,

$$\phi(\tilde{r}, y, \Omega) = \sum_{l, \vec{m}} \tilde{\phi}_{l, \vec{m}}(\tilde{r}, y) Y_{l, \vec{m}}(\Omega). \quad (2.15)$$

For the remainder of this discussion we consider a fixed  $l$  mode. Note that the reality of  $\phi$  imposes the condition

$$\tilde{\phi}_{l, \vec{m}}^* = \tilde{\phi}_{l, -\vec{m}}. \quad (2.16)$$

Thus if we expand field in the usual spherical harmonics then we should write Equation (2.14) as

$$\int D[X] \exp \left( -S_{\text{CFT}}[X] + \mu \sum_{\vec{m}} \int d^d y \sqrt{g_d} \tilde{\phi}_{b, \vec{m}} \mathcal{V}_{\vec{m}} \right) = \int D[\phi] e^{-S_{\text{AdS}}[\phi]} \Big|_{\tilde{\phi}_{\vec{m}}(\tilde{r}_b, y) \equiv \tilde{\phi}_{b, \vec{m}}(y)}. \quad (2.17)$$

Reality of the action requires

$$\hat{\mathcal{V}}_{\vec{m}}^\dagger(y) = \hat{\mathcal{V}}_{-\vec{m}}(y). \quad (2.18)$$

At leading order we can replace the gravity path integral by the classical action evaluated with the given boundary values of the gravity fields,

$$\int D[\phi] e^{-S_{\text{AdS}}[\phi]} \Big|_{\phi(r_b) = \phi_b} = e^{-S_{\text{AdS}}[\phi^{\text{cl.}}]}. \quad (2.19)$$

Then the 2-point function in the CFT is given by

$$\langle \hat{\mathcal{V}}_{\vec{m}}(y_1) \hat{\mathcal{V}}_{\vec{m}'}(y_2) \rangle = \frac{1}{\mu^2} \frac{\delta}{\delta \tilde{\phi}_{b,\vec{m}}(y_1)} \frac{\delta}{\delta \phi_{b,\vec{m}'}(y_2)} (-S_{\text{AdS}}). \quad (2.20)$$

Let us now define the normalizations of  $\phi$  and  $\hat{\mathcal{V}}$ . We consider a minimal scalar field for concreteness, though our computations should be extendable to other supergravity fields with no difficulty. The gravity action is

$$S_{\text{AdS}} = \frac{1}{16\pi G_D} \int d^D x \sqrt{g} \left( \frac{1}{2} \partial \phi \partial \phi \right), \quad (2.21)$$

where  $D = d + 1 + p$  is the dimension of the spacetime ( $D = 10$  for string theory,  $D = 11$  for M theory, and we have  $D = 6$  for the D1D5 system after we reduce on a compact  $T^4$ ). In region (iii), the spacetime has the form

$$AdS_{d+1} \times S^p \times \mathcal{M}. \quad (2.22)$$

For cases like the D1D5 system we have in addition a compact 4-manifold  $\mathcal{M} = T^4$  or  $\mathcal{M} = K3$ . We take  $\phi$  to be a zero mode on  $\mathcal{M}$  and dimensionally reduce on  $\mathcal{M}$ , so that we again have a space  $AdS_3 \times S^3$ , with  $G$  now the 6-d Newton's constant.

If the spherical harmonics are normalized such that,

$$\int d\Omega |Y_{l,\vec{m}}(\Omega)|^2 = 1, \quad (2.23)$$

then dimensionally reducing on  $S^p$  yields

$$S_{\text{AdS}} = \frac{R_s^p}{16\pi G_D} \sum_{\vec{m}} \int d^{d+1} y \sqrt{g_{d+1}} \left[ \frac{1}{2} |\partial \tilde{\phi}_{\vec{m}}|^2 + \frac{1}{2} m^2 |\tilde{\phi}_{\vec{m}}|^2 \right]. \quad (2.24)$$

The  $l$ -dependent mass,  $m$ , comes from

$$m^2 = \frac{\Lambda}{R_s^2}, \quad \triangle_p Y(\Omega) = -\Lambda Y(\Omega), \quad \Lambda = l(l + p - 1). \quad (2.25)$$

The CFT lives on the surface  $\tilde{r} = \tilde{r}_b$ . The metric on this surface is (from (2.11))

$$ds^2 = \left( \frac{\tilde{r}_b}{R_s} \right)^2 \sum_{i=1}^d dy_i dy_i. \quad (2.26)$$

We choose the normalization of the operators  $\hat{\mathcal{V}}$  by requiring the 2-point function to have the short distance expansion  $\sim \frac{1}{(\text{distance})^{2\Delta}}$ :

$$\langle \hat{\mathcal{V}}_{\vec{m}}(y_1) \hat{\mathcal{V}}_{\vec{m}'}(y_2) \rangle = \frac{\delta_{\vec{m}+\vec{m}',0}}{\left[ \left( \frac{\tilde{r}_b}{R_s} \right) |y_1 - y_2| \right]^{2\Delta}}. \quad (2.27)$$

Following [8], we define the boundary-to-bulk propagator which gives the value of  $\phi$  in the  $AdS$  region given its value on the boundary at  $r_b$ :

$$\tilde{\phi}_{\vec{m}}(\tilde{r}, y) = \int K(\tilde{r}, y; y') \tilde{\phi}_{b, \vec{m}}(y') \sqrt{g_d} d^d y', \quad (2.28)$$

where  $\sqrt{g_d} d^d y'$  is the volume element on the metric (2.26) on the boundary. We have

$$K(\tilde{r}, y; y') = \frac{R_{\text{AdS}}^{2\Delta-d}}{\tilde{r}_b^\Delta \pi^{\frac{d}{2}}} \frac{\Gamma(\Delta)}{\Gamma(\Delta - \frac{d}{2})} \left[ \frac{\tilde{r}}{R_{\text{AdS}}^2 + \frac{\tilde{r}^2}{R_s^2} |y - y'|^2} \right]^\Delta, \quad (2.29)$$

where

$$\Delta = \frac{1}{2}(d + \sqrt{d^2 + 4m^2 R_{\text{AdS}}^2}) = (l + p - 1) \frac{d}{p - 1}. \quad (2.30)$$

We then have

$$\frac{\delta}{\delta \tilde{\phi}_{b, \vec{m}}(y_1)} \frac{\delta}{\delta \phi_{b, \vec{m}'}(y_2)} (-S_{\text{AdS}}) = -\frac{R_s^p}{16\pi G_D} \delta_{\vec{m} + \vec{m}', 0} \left( \frac{2\Delta - d}{\Delta} \right) \partial_{\tilde{r}} K(\tilde{r}, y_1; y_2) \left( \sqrt{g^{\tilde{r}\tilde{r}}} \Big|_{\tilde{r}=\tilde{r}_b} \right), \quad (2.31)$$

where the extra factor  $\frac{2\Delta-d}{\Delta}$  comes from taking care with the limit  $\tilde{r}_b \rightarrow \infty$  when using the kernel (2.29) [46].

Putting (2.27) and (2.31) into (2.20) we get

$$\mu = \left[ \frac{R_s^{2\Delta-(d+1)+p} \left( \frac{d}{p-1} \right)^{2\Delta-(d+1)}}{16\pi G_D} \frac{(2\Delta-d)}{\pi^{\frac{d}{2}}} \frac{\Gamma(\Delta)}{\Gamma(\Delta - \frac{d}{2})} \right]^{\frac{1}{2}}. \quad (2.32)$$

## 2.2 The outer region

The wave equation for the minimal scalar is

$$\square \phi = 0. \quad (2.33)$$

We write

$$\phi = h(r) Y(\Omega) e^{-iEt} e^{i\vec{\lambda} \cdot \vec{y}}, \quad (2.34)$$

getting the solution

$$h = \frac{1}{r^{\frac{p-1}{2}}} \left[ C_1 J_{l+\frac{p-1}{2}} \left( \sqrt{E^2 - \lambda^2} r \right) + C_2 J_{-l-\frac{p-1}{2}} \left( \sqrt{E^2 - \lambda^2} r \right) \right]. \quad (2.35)$$

Note that we can define a CFT in the  $AdS$  region only if the excitations in the  $AdS$  region decouple to leading order from the flat space part of the geometry [10]. Such an approximate decoupling happens for quanta with energies  $E \ll 1/R_s$ : waves incident from infinity with wavelengths much longer than the  $AdS$  curvature scale almost completely reflect off the 'neck' region and there is only a small probability of absorption into the  $AdS$  part of the geometry.



Correspondingly, waves with such energies trapped in the  $AdS$  region have only a small rate of leakage to flat space. Thus we work throughout this paper with the assumption

$$E \ll \frac{1}{R_s}. \quad (2.36)$$

With this, we find that to leading order the wave in the outer region has the  $C_2 \approx 0$  [47]:

$$h \approx C_1 \frac{1}{\Gamma(l + \frac{p+1}{2})} \left[ \frac{\sqrt{E^2 - \lambda^2}}{2} \right]^{l + \frac{p-1}{2}} r^l. \quad (2.37)$$

A general wave is a superposition of different modes of the form (2.34). We wish to extract a given spherical harmonic from this wave, so that we can couple it to the appropriate vertex operator of the CFT. Define

$$[\partial^l \phi]^{l, \vec{m}} = Y_{l, \vec{m}}^{k_1 k_2 \dots k_l} \partial_{k_1} \partial_{k_2} \dots \partial_{k_l} \phi, \quad (2.38)$$

where the above differential operator is normalized such that<sup>2</sup>

$$Y_{l, \vec{m}}^{k_1 k_2 \dots k_l} \partial_{k_1} \partial_{k_2} \dots \partial_{k_l} [r^{l'} Y_{l', \vec{m}'}(\Omega)] = \delta_{ll'} \delta_{\vec{m}, \vec{m}'}. \quad (2.39)$$

Thus the required angular component of  $\phi$  at small  $r$  satisfies

$$\phi \approx [\partial^l \phi] \Big|_{r \rightarrow 0}^{l, \vec{m}} r^l Y_{l, \vec{m}}(\Omega). \quad (2.40)$$

### 2.3 The intermediate region

In the ‘neck’ region we can set  $E, \lambda$  to zero in solving the wave equation, since we assume that we are at low energies and momenta so the wavelength is large compared to the size of the intermediate region. Thus the  $E, \lambda$  terms do not induce oscillations of the waveform in the limited domain of the intermediate region.

With this approximation we now have to solve the wave equation in the intermediate region. From this solution, we need the following information to construct our full solution. Suppose in the outer part of the intermediate region  $r \sim Q_{\max}^{\frac{1}{p-1}}$  we have the solution

$$\phi \approx r^l. \quad (2.41)$$

Evolved to the inner part of the intermediate region  $r \ll Q_{\max}^{\frac{1}{p-1}}$ , we have a form given by  $AdS$  physics:

$$\phi \approx b_l \tilde{r}^{\Delta - d}. \quad (2.42)$$

These two numbers,  $b_l, \Delta$ , give the information we need about the effect of the intermediate region on the wavefunction.  $\Delta$  is known from the CFT, while the number  $b_l$  appears in our

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<sup>2</sup>See Appendix B for more details.

final expression for the emission amplitude as representing the physics of the intermediate region which connects the *AdS* region to flat infinity.

We now note that for the case that we work with, the minimally coupled scalar, we can in fact write down the values of  $b_l, \Delta$ . The wave equation for a minimally coupled scalar in the background (2.1) with the ansatz (2.34) takes the form

$$H^{\frac{2(d+p-1)}{d(p-1)}}(E^2 - \lambda^2)r^2h(r) + \frac{1}{r^{p-2}}\partial_r(r^p\partial_rh(r)) - l(l+p-1)h(r) = 0. \quad (2.43)$$

The term

$$H^{\frac{2(d+p-1)}{d(p-1)}}r^2 \quad (2.44)$$

is bounded in the ‘neck’ region (2.6). Therefore, assuming small  $E, \lambda$ , the wave equation in the neck is

$$\frac{1}{r^{p-2}}\partial_r(r^p\partial_rh(r)) - l(l+p-1)h(r) = 0. \quad (2.45)$$

This has the solution

$$h(r) = Ar^l + Br^{-l-p+1}. \quad (2.46)$$

Thus we see that if we use the coordinate  $r$  throughout the intermediate region, then there is no change in the functional form of  $\phi$  as we pass through the intermediate region. We are interested in the  $r^l$  solution, so we set  $B = 0$ . We must now write this in terms of the coordinate  $\tilde{r}$  appropriate for the *AdS* region. First consider the case  $d = p - 1$  which holds for the D3 and D1D5 cases. Then we see that

$$\tilde{r} = r, \quad b_l = 1. \quad (2.47)$$

Now consider  $d \neq p - 1$ . From (2.10) and (2.30) we get

$$b_l = R_s^{l(1-\frac{d}{p-1})}. \quad (2.48)$$

In general the scalars are not ‘minimal’; i.e., they can have couplings to the gauge fields present in the geometry. Then  $b_l$  needs to be computed by looking at the appropriate wave equation. Examples of such non-minimal scalars are ‘fixed’ scalars discussed in [48].

To summarize, we find that the change of the waveform through the intermediate region is given by  $b_l, \Delta$ . The wave at the boundary  $\tilde{r} = \tilde{r}_b$  is then

$$\tilde{\phi}_{b,\vec{m}}(y) = b_l \tilde{r}_b^{\Delta-d} [\partial^l \phi(y)] \Big|_{r=0}^{l,\vec{m}}. \quad (2.49)$$

## 2.4 The interaction

We can break the action of the full problem into three parts

$$S_{\text{total}} = S_{\text{CFT}} + S_{\text{outer}} + S_{\text{int}}, \quad (2.50)$$

where the contribution of the interaction between the CFT and the outer asymptotically flat region,  $S_{\text{int}}$ , vanishes in the strict decoupling limit. We work in the limit where the interaction is small but nonvanishing, to first order in the interaction.

The coupling of the external wave *at*  $r_b$  to the CFT is given by the interaction

$$S_{\text{int}}^l = -\mu \sum_{\vec{m}} \int \sqrt{g_d} d^d y \tilde{\phi}_{b,\vec{m}}(y) \hat{\mathcal{V}}_{l,\vec{m}}(y). \quad (2.51)$$

If we want to *directly* couple to the modes in the *outer region*, then we can incorporate the intermediate region physics into  $S_{\text{int}}$  by writing

$$S_{\text{int}}^l = -c_l \sum_{\vec{m}} \int \sqrt{g_d} d^d y [\partial^l \phi(y)] \Big|_{r=0}^{l,\vec{m}} \hat{\mathcal{V}}_{l,\vec{m}}(y), \quad (2.52)$$

where

$$c_l = \mu b_l \tilde{r}_b^{\Delta-d}. \quad (2.53)$$

This is the general action connecting modes in the AdS/CFT with the modes in the asymptotically flat space. We focus specifically on (first-order) emission processes, but one can consider more general interactions (absorption, scattering, etc.).

## 2.5 Emission

Suppose we have an excited state in the ‘cap’ region,  $|i\rangle$ , and the vacuum in the outer region,  $|\emptyset\rangle_{\text{outer}}$ . Because of the coupling (2.53), a particle can be emitted and escape to infinity, changing the state in the cap to a lower-energy state  $|f\rangle$ , and leaving a 1-particle state in the outer region,  $|E, l, \vec{m}, \vec{\lambda}\rangle_{\text{outer}}$ . We wish to compute the rate for this emission,  $\Gamma$ . We can write the total amplitude for this process as

$$\mathcal{A} = \left( {}_{\text{outer}} \langle E, l, \vec{m}, \vec{\lambda} | \langle f | \right) i S_{\text{int}} \left( |i\rangle |\emptyset\rangle_{\text{outer}} \right). \quad (2.54)$$

We quantize the field  $\phi$  in the outer region as

$$\begin{aligned} \hat{\phi} = \sqrt{\frac{16\pi G_D}{2V_y}} \sum_{\vec{\lambda}, l, \vec{m}} \int_0^\infty dE \frac{J_{l+\frac{p+1}{2}}(\sqrt{E^2-\lambda^2} r)}{r^{\frac{p-1}{2}}} \Big[ \hat{a}_{E,l,\vec{m},\lambda} Y_{l,\vec{m}} e^{i(\vec{\lambda}\cdot\vec{y}-Et)} \\ + (\hat{a}_{E,l,\vec{m},\lambda})^\dagger Y_{l,\vec{m}}^* e^{-i(\vec{\lambda}\cdot\vec{y}-Et)} \Big], \end{aligned} \quad (2.55)$$

where

$$[\hat{a}_{E,l,\vec{m},\lambda}, (\hat{a}_{E',l',\vec{m}',\lambda'})^\dagger] = \delta_{ll'} \delta_{\vec{m},\vec{m}'} \delta_{\lambda\lambda'} \delta(E-E'). \quad (2.56)$$

Using the asymptotic behavior,

$$J_\nu(z) \approx \frac{1}{\Gamma(\nu+1)} \left(\frac{z}{2}\right)^\nu, \quad (2.57)$$

we find that

$$\begin{aligned} [\partial^l \hat{\phi}] \Big|_{r=0}^{\vec{m}} = \sqrt{\frac{16\pi G_D}{2V_y}} \sum_{\vec{\lambda}} \int_0^\infty dE \frac{(\frac{\sqrt{E^2-\lambda^2}}{2})^{l+\frac{p-1}{2}}}{\Gamma(l+\frac{p+1}{2})} \Big[ \hat{a}_{E,l,\vec{m},\lambda} e^{i(\vec{\lambda}\cdot\vec{y}-Et)} \\ + (\hat{a}_{E,l,-\vec{m},\lambda})^\dagger e^{-i(\vec{\lambda}\cdot\vec{y}-Et)} \Big]. \end{aligned} \quad (2.58)$$

Using the coupling  $c_l$  from (2.53) we get the interaction Lagrangian

$$S_{\text{int}} = -\mu b_l \tilde{r}_b^{\Delta-d} \sqrt{\frac{16\pi G_D}{2V_y}} \sum_{\vec{\lambda}, \vec{m}} \int \sqrt{g_d} d^d y \int_0^\infty dE \frac{\left(\frac{\sqrt{E^2-\lambda^2}}{2}\right)^{l+\frac{p-1}{2}}}{\Gamma(l+\frac{p+1}{2})} \left[ \hat{a}_{E,l,\vec{m},\lambda} e^{i(\vec{\lambda}\cdot\vec{y}-Et)} + (\hat{a}_{E,l,-\vec{m},\lambda})^\dagger e^{-i(\vec{\lambda}\cdot\vec{y}-Et)} \right] \hat{\mathcal{V}}_{l,\vec{m}}(t, \vec{y}). \quad (2.59)$$

We can pull out the  $(t, \vec{y})$  dependence of the CFT part of the amplitude by writing

$$\langle f | \hat{\mathcal{V}}(t, \vec{y}) | i \rangle = e^{-iE_0 t + i\vec{\lambda}_0 \cdot \vec{y}} \langle f | \hat{\mathcal{V}}(t=0, \vec{y}=0) | i \rangle, \quad (2.60)$$

where  $E_0$  and  $\lambda_0$  can be determined from the initial and final states of the CFT,  $|i\rangle$  and  $|f\rangle$ . We also work in the case where the initial and final CFT states select out a single  $l, m$  mode in the interaction, whose indices have been suppressed. The CFT lives on a space of (coordinate) volume  $V_y$ . When computing CFT correlators we work in a ‘unit-sized’ space with volume  $(2\pi)^{d-1}$ . Scaling the operator  $\hat{\mathcal{V}}$  we have

$$\langle f | \hat{\mathcal{V}}(t=0, \vec{y}=0) | i \rangle = \left[ \frac{(2\pi)^{d-1}}{(\frac{\tilde{r}_b}{R_s})^{d-1} V_y} \right]^{\frac{\Delta}{d-1}} \langle f | \hat{\mathcal{V}}(t=0, \vec{y}=0) | i \rangle_{\text{unit}}. \quad (2.61)$$

Rotating back to Lorentzian signature, the amplitude for emission of a quantum from an excited state of the CFT is

$$\begin{aligned} \mathcal{A} = & -i\mu b_l \tilde{r}_b^{\Delta-d} \sqrt{\frac{16\pi G_D}{2V_y}} \frac{\left(\frac{\sqrt{E^2-\lambda^2}}{2}\right)^{l+\frac{p-1}{2}}}{\Gamma(l+\frac{p+1}{2})} \left[ \frac{(2\pi)^{d-1}}{(\frac{\tilde{r}_b}{R_s})^{d-1} V_y} \right]^{\frac{\Delta}{d-1}} \langle f | \hat{\mathcal{V}}_{l,\vec{m}}(t=0, \vec{y}=0) | i \rangle_{\text{unit}} \\ & \times \int_0^T dt \int \left(\frac{\tilde{r}_b}{R_s}\right)^d d^{d-1} y e^{-i(E_0-E)t} e^{i(\vec{\lambda}_0-\vec{\lambda})\cdot\vec{y}}. \end{aligned} \quad (2.62)$$

The amplitude gives the emission rate in a straightforward calculation:

$$\begin{aligned} \frac{d\Gamma}{dE} &= \lim_{T \rightarrow \infty} \frac{|\mathcal{A}|^2}{T} \\ &= (2\pi)^{2\Delta+1} \frac{R_s^{4\Delta-3d+p-1} \left(\frac{d}{p-1}\right)^{2\Delta-(d+1)}}{V_y^{\frac{2\Delta-d+1}{d-1}}} \left[ \frac{(2\Delta-d)}{2\pi^{\frac{d}{2}}} \frac{\Gamma(\Delta)}{\Gamma(\Delta-\frac{d}{2})} \right] |b_l|^2 \left[ \frac{1}{\Gamma(l+\frac{p+1}{2})} \right]^2 \\ &\quad \times \left( \frac{E^2-\lambda^2}{4} \right)^{l+\frac{p-1}{2}} \left| \langle 0 | \hat{\mathcal{V}}_{l,-\vec{m}}(0) | 1 \rangle_{\text{unit}} \right|^2 \delta_{\vec{\lambda}, \vec{\lambda}_0} \delta(E-E_0). \end{aligned} \quad (2.63)$$

From this expression, we see that in the strict decoupling limit where  $ER_s \rightarrow \infty$  this rate vanishes as expected.

We have derived the above result for a general CFT and its corresponding brane geometry. In the remainder of the paper we work with the D1D5 system. For a minimal scalar in the D1D5 geometry we have

$$d=2, \quad p=3 \quad \Delta_{\text{tot.}} = l+2 \quad b_l=1 \quad R_s = (Q_1 Q_5)^{\frac{1}{4}} \quad V_y = 2\pi R. \quad (2.64)$$

Plugging in, we reduce the decay rate formula to the form

$$\frac{d\Gamma}{dE} = \frac{2\pi}{2^{2l+1} l!^2} \frac{(Q_1 Q_5)^{l+1}}{R^{2l+3}} (E^2 - \lambda^2)^{l+1} |\langle 0 | \hat{\mathcal{V}} | 1 \rangle_{\text{unit}}|^2 \delta_{\lambda, \lambda_0} \delta(E - E_0). \quad (2.65)$$

### 3. The D1D5 CFT at the orbifold point

#### 3.1 The CFT

Consider type IIB string theory, compactified as

$$M_{9,1} \rightarrow M_{4,1} \times S^1 \times T^4. \quad (3.1)$$

Wrap  $N_1$  D1 branes on  $S^1$ , and  $N_5$  D5 branes on  $S^1 \times T^4$ . The bound state of these branes is described by a field theory. We think of the  $S^1$  as being large compared to the  $T^4$ , so that at low energies we look for excitations only in the direction  $S^1$ . This low energy limit gives a conformal field theory (CFT) on the circle  $S^1$ .

We can vary the moduli of string theory (the string coupling  $g$ , the shape and size of the torus, the values of flat connections for gauge fields etc.). These changes move us to different points in the moduli space of the CFT. It has been conjectured that we can move to a point called the ‘orbifold point’ where the CFT is particularly simple [49, 50, 51, 52, 53, 54, 55, 56]. At this orbifold point the CFT is a 1+1 dimensional sigma model. The 1+1 dimensional base space is spanned by  $(y, t)$ , where

$$0 \leq y < 2\pi R \quad (3.2)$$

is a coordinate along the  $S^1$ , and  $t$  is the time of the 10-d string theory. For our CFT computations, we rotate time to Euclidean time, and also use scaled coordinates  $(\sigma, \tau)$  where the space direction of the CFT has length  $2\pi$ :

$$\tau = i \frac{t}{R} \quad \sigma = \frac{y}{R}. \quad (3.3)$$

Moreover, we find it convenient to work in the complex plane with coordinates  $(z, \bar{z})$  defined by the exponential map,

$$z = e^{\tau + i\sigma} \quad \bar{z} = e^{\tau - i\sigma}. \quad (3.4)$$

We continue back to Lorentzian signature at the end.

The target space of the sigma model is the ‘symmetrized product’ of  $N_1 N_5$  copies of  $T^4$ ,

$$(T_4)^{N_1 N_5} / S_{N_1 N_5}, \quad (3.5)$$

with each copy of  $T^4$  giving 4 bosonic excitations  $X^1, X^2, X^3, X^4$ . It also gives 4 fermionic excitations, which we call  $\psi^1, \psi^2, \psi^3, \psi^4$  for the left movers, and  $\bar{\psi}^1, \bar{\psi}^2, \bar{\psi}^3, \bar{\psi}^4$  for the right movers. The fermions can be antiperiodic or periodic around the  $\sigma$  circle. If they are antiperiodic on the  $S^1$  we are in the Neveu-Schwarz (NS) sector, and if they are periodic on the  $S^1$  we are in the Ramond (R) sector<sup>3</sup>.

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<sup>3</sup>The periodicities flip when mapping to the complex plane because of a Jacobian factor.

### 3.1.1 Twist operators

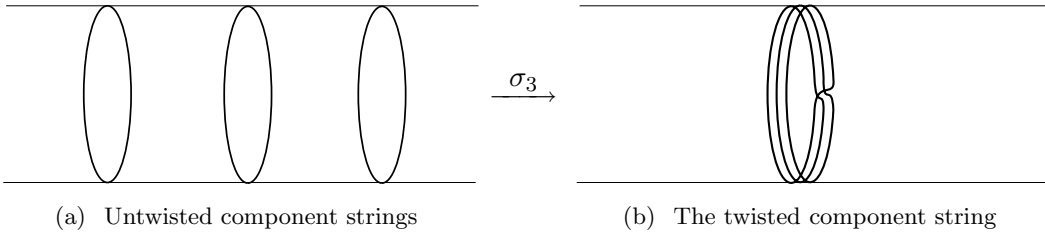
Since we orbifold by the symmetric group  $S_{N_1 N_5}$ , we generate ‘twist sectors’, which can be obtained by acting with ‘twist operators’  $\sigma_n$  on an untwisted state. Suppose we insert a twist operator at a point  $z$  in the base space. As we circle the point  $z$ , different copies of  $T^4$  get mapped into each other. Let us denote the copy number by a subscript  $a = 1, 2, \dots n$ . The twist operator is labeled by the permutation it generates. For instance, every time one circles the twist operator

$$\sigma_{(123\dots n)}, \quad (3.6)$$

the fields  $X_{(a)}^i$  get mapped as

$$X_{(1)}^i \rightarrow X_{(2)}^i \rightarrow \dots \rightarrow X_{(n)}^i \rightarrow X_{(1)}^i, \quad (3.7)$$

and the other copies of  $X_{(a)}^i$  are unchanged. We have a similar action on the fermionic fields. We depict this ‘twisting’ in Fig. 2. Each set of linked copies of the CFT is called one ‘component string’.



**Figure 2:** The twist operator  $\sigma_3$ . Each loop represents a ‘copy’ of the CFT wrapping the  $S^1$ . The twist operator joins these copies into one single copy of the CFT living on a circle of three times the length of the original circle.

We often abbreviate a twist operator like the one in Equation (3.6) with  $\sigma_n$  for simplicity (we have to give the indices involved in the permutation explicitly when we use  $\sigma_n$  in a correlator). We call these operators,  $\sigma_n$ , ‘bare twists’ to distinguish them from the more relevant operators for our purpose, which have additional ‘charges’ added to the ‘bare twist’ forming operators that are chiral primaries for the supersymmetric CFT.

### 3.1.2 Symmetries of the CFT

The D1D5 CFT has  $(4, 4)$  supersymmetry, which means that we have  $\mathcal{N} = 4$  supersymmetry in both the left and right moving sectors. This leads to a superconformal  $\mathcal{N} = 4$  symmetry in both the left and right sectors, generated by operators  $L_n, G_r^\pm, J_n^a$  for the left movers and  $\bar{L}_n, \bar{G}_r^\pm, \bar{J}_n^a$  for the right movers. The expressions for the left generators in terms of the  $X^i, \psi^i$  are given in Equation (A.6). The OPEs are given in Equation (A.15), and the (anti-)commutation relations between modes is given in Equation (A.18).

Each  $\mathcal{N} = 4$  algebra has an internal R symmetry group  $SU(2)$ ,<sup>4</sup> so there is a global symmetry group  $SU(2)_L \times SU(2)_R$ . We denote the quantum numbers in these two  $SU(2)$  groups as

$$SU(2)_L : (j, m); \quad SU(2)_R : (\bar{j}, \bar{m}). \quad (3.8)$$

In the geometrical setting of the CFT, this symmetry arises from the rotational symmetry in the 4 space directions of  $M_{4,1}$  in Equation (3.1),

$$SO(4)_E \simeq SU(2)_L \times SU(2)_R. \quad (3.9)$$

Here the subscript  $E$  stands for ‘external’, which denotes that these rotations are in the noncompact directions. These quantum numbers therefore give the angular momentum of quanta in the gravity description. We have another  $SO(4)$  symmetry in the four directions of the  $T^4$ . This symmetry we call  $SO(4)_I$  (where  $I$  stands for ‘internal’). This symmetry is broken by the compactification of the torus, but at the orbifold point it still provides a useful organizing principle. We write

$$SO(4)_I \simeq SU(2)_1 \times SU(2)_2. \quad (3.10)$$

We use spinor indices  $\alpha, \dot{\alpha}$  for  $SU(2)_L$  and  $SU(2)_R$  respectively. We use spinor indices  $A, \dot{A}$  for  $SU(2)_1$  and  $SU(2)_2$  respectively.

The 4 real fermions of the left sector can be grouped into complex fermions  $\psi^{\alpha A}$  with the reality constraint

$$(\psi^{\alpha \dot{A}})^\dagger = -\epsilon_{\alpha\beta} \epsilon_{\dot{A}\dot{B}} \psi^{\beta \dot{B}} = -\psi_{\alpha \dot{A}}. \quad (3.11)$$

The right fermions have indices  $\bar{\psi}^{\dot{\alpha} \dot{A}}$  with a similar reality constraint. The bosons  $X^i$  are a vector in the  $T^4$ . Thus they have no charge under  $SU(2)_L$  or  $SU(2)_R$  and are given by

$$[X]^{\dot{A}A} = X^i (\sigma^i)^{\dot{A}A}. \quad (3.12)$$

where  $\sigma^i, i = 1, \dots, 4$  are the three Pauli matrices and the identity. (The notations described here are explained in full detail in Appendix A.)

### 3.2 States of the CFT

The CFT arising from the D1D5 brane bound state is in the Ramond (R) sector. One can understand this because the periodicities of the fermions around the  $S^1$  are inherited from the behavior of fermionic supergravity fields around the  $S^1$  in (3.1). These supergravity fields must be taken to be periodic, since otherwise we would generate a nonzero vacuum energy in our spacetime and the metric far from the branes would not be flat.

Even though the physical CFT problem is in the R sector, we find it convenient to map our R sector states to the NS sector using *spectral flow*, which simplifies the calculation. Using spectral flow we can relate one calculation to a whole family of related processes. Thus let us first look at states in the NS sector.

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<sup>4</sup>In fact, the full R symmetry group of the  $\mathcal{N} = 4$  algebra is  $SO(4)$ ; however, the other  $SU(2)$  does not have a current associated with it within the algebra.

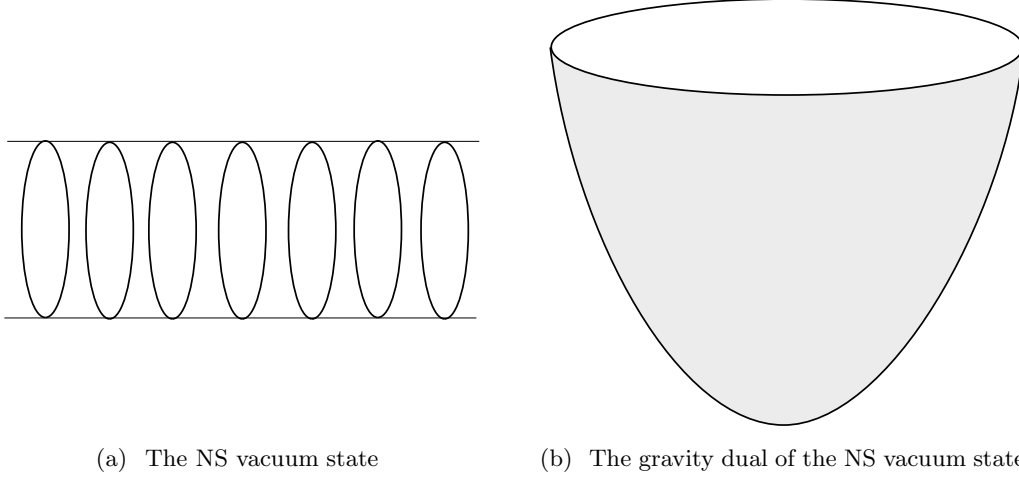
### 3.3 States in the NS sector

There are two general pieces of information needed to describe the states of the orbifold CFT. First we have to look at the ‘twist sector’; i.e., note which copies of the CFT are linked to which copies as we go around the  $S^1$ . The second thing we have to look at are the bosonic and fermionic excitations in the given twist sector.

The simplest sector is the ‘untwisted sector’, where the copies are all delinked from each other. Let us take the state in this sector with *no* excitations. This state is depicted in Fig. 3(a). It is the NS vacuum, and has quantum numbers

$$|\emptyset\rangle_{NS} : \quad h = \bar{h} = 0; \quad j = m = \bar{j} = \bar{m} = 0. \quad (3.13)$$

The gravity dual of this state is ‘global AdS’, depicted in Fig. 3(b).



**Figure 3:** (a) The NS vacuum state in the CFT and (b) its gravity dual, which is global  $AdS$ . The NS vacuum is the simplest possible state having no twists, no excitations, and no base spin.

Now consider excited states. The simplest states are ‘chiral primaries’, which have ‘dimension=charge’:

$$h = m, \quad \bar{h} = \bar{m}. \quad (3.14)$$

We can start with a chiral primary and act with the ‘anomaly-free subalgebra’ of the chiral algebra to make descendants. This subalgebra is spanned by

$$\{L_{-1}, L_0, L_1, G_{\pm\frac{1}{2}}^{\pm}, J_0^a\}. \quad (3.15)$$

All states obtained by starting with a chiral primary and applying operators (3.15) correspond to supergravity excitations in the dual gravity theory. Other states correspond to ‘stringy’ states. We are interested in the emission of a supergravity quantum from the CFT state, so let us look at such states in more detail.



### 3.3.1 The basic chiral primary operators $\sigma_{l+1}^0$

Let us recall the construction of chiral primary operators introduced in [57].

Start with the NS vacuum state  $|\emptyset\rangle_{NS}$ , which is in the completely untwisted sector, where all the component strings are ‘singly wound’. The gravity dual is global AdS.

Now suppose in this gravity dual we want to add one supergravity quantum carrying angular momentum  $\frac{l}{2}$  in each of the two factors of  $SU(2)_L \times SU(2)_R$  (scalars must have  $j = \bar{j}$ ). We take a set of  $l+1$  copies of the CFT and join them by using a twist operator  $\sigma_{l+1}$  into one ‘multiply wound’ component string. Let us take the lowest energy state in this twist sector. This state has dimensions [57]

$$h = \bar{h} = \frac{c}{24} \left[ (l+1) - \frac{1}{(l+1)} \right], \quad (3.16)$$

but it does not yet have any charge, so it is not a chiral primary; it has more dimension than charge. The current operator  $J^+$  carries positive charge, and we can apply contour integrals of  $J^+$  to our state to raise its charge. Since all operators in the theory have  $h > |m|$ , one might think that we cannot reach a chiral primary with  $h = j = m$  if we start with the state (3.16); however, on the twisted component string we can apply *fractional* modes  $J_{\frac{k}{l+1}}^a$  of the current operators, because any contour integral around the twist operator insertion has to close only after going around the insertion  $l+1$  times.

Before we apply these fractionally-moded current operators, there is one more point to note. For the case of  $l+1$  even one finds that the twist operator  $\sigma_n$  yields antiperiodic boundary conditions for the fermion field when we traverse around the twist insertion  $l+1$  times. Since we wanted the fermion field to return to itself after going  $l+1$  times around the insertion, we must insert a ‘spin field’ to change the periodicity of these fermions. The construction of these spin fields was explained in detail in [57], but for now we just denote the twist with spin field insertions (for both left and right fermions) as  $(S_{l+1}^+ \bar{S}_{l+1}^+ \sigma_{l+1})$ . With this notation we find that the chiral primaries are given by<sup>5</sup>

$$\sigma_{l+1}^0 = \begin{cases} J_{-\frac{l-1}{l+1}}^+ J_{-\frac{l-3}{l+1}}^+ \cdots J_{-\frac{1}{l+1}}^+ \bar{J}_{-\frac{l-1}{l+1}}^+ \bar{J}_{-\frac{l-3}{l+1}}^+ \cdots \bar{J}_{-\frac{1}{l+1}}^+ \sigma_{l+1} & (l+1) \text{ odd} \\ J_{-\frac{l-1}{l+1}}^+ J_{-\frac{l-3}{l+1}}^+ \cdots J_{-\frac{2}{l+1}}^+ \bar{J}_{-\frac{l-1}{l+1}}^+ \bar{J}_{-\frac{l-3}{l+1}}^+ \cdots \bar{J}_{-\frac{2}{l+1}}^+ (S_{l+1}^+ \bar{S}_{l+1}^+ \sigma_{l+1}) & (l+1) \text{ even.} \end{cases} \quad (3.17)$$

This construction generates chiral primaries with dimensions and charges

$$\sigma_{l+1}^0 : \quad h = m = \frac{l}{2}, \quad \bar{h} = \bar{m} = \frac{l}{2}. \quad (3.18)$$

Note that  $J^+ \sim \psi^{+\dot{1}} \psi^{+\dot{2}}$ , and the current operators in (3.17) fill up the left and right moving Fermi seas up to a ‘Fermi level’ (here we write only the left sector)

$$\sigma_{l+1}^0 \sim \begin{cases} \psi_{-\frac{l-1}{2(l+1)}}^{+\dot{1}} \psi_{-\frac{l-1}{2(l+1)}}^{+\dot{2}} \psi_{-\frac{l-3}{2(l+1)}}^{+\dot{1}} \psi_{-\frac{l-3}{2(l+1)}}^{+\dot{2}} \cdots \psi_{-\frac{1}{2(l+1)}}^{+\dot{1}} \psi_{-\frac{1}{2(l+1)}}^{+\dot{2}} \sigma_{l+1} & (l+1) \text{ odd} \\ \psi_{-\frac{l-1}{2(l+1)}}^{+\dot{1}} \psi_{-\frac{l-1}{2(l+1)}}^{+\dot{2}} \psi_{-\frac{l-3}{2(l+1)}}^{+\dot{1}} \psi_{-\frac{l-3}{2(l+1)}}^{+\dot{2}} \cdots \psi_{-\frac{1}{2(l+1)}}^{+\dot{1}} \psi_{-\frac{1}{2(l+1)}}^{+\dot{2}} (S_{l+1}^+ \sigma_{l+1}) & (l+1) \text{ even.} \end{cases} \quad (3.19)$$

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<sup>5</sup>Chiral primary operators can also be written using alternative descriptions which make use of bosonized fermions (see for example [55, 11]).

### 3.3.2 Additional chiral primaries

Above we described the construction of the simplest chiral primary  $\sigma_{l+1}^0$ . We can make additional chiral primaries as follows:

- (i) The next available fermion level for the fermion  $\psi^{+1}$  in (3.19) is  $\psi_{-\frac{1}{2}}^{+1}$ . If we fill this level, we raise both dimension and charge by  $\frac{1}{2}$ , so we get another chiral primary.
- (ii) We can do the same with the fermion  $\psi^{+2}$ .
- (iii) We can add both fermions  $\psi^{+1}, \psi^{+2}$ , which is equivalent to an application of  $J_{-1}^+$ .

Of course we can make analogous excitations to the right sector as well. This gives a total of  $4 \times 4 = 16$  chiral primaries in a given twist sector. This exhausts all the possible chiral primaries for this system.

### 3.3.3 Anti-chiral primaries

We define anti-chiral primaries as states with

$$h = -m, \quad \bar{h} = -\bar{m}. \quad (3.20)$$

To construct these states, we again start with a twist operator  $\sigma_{l+1}$  and then apply modes of  $J^-$  instead of  $J^+$ . Proceeding in the same way as for chiral primaries, we get the anti-chiral primary (denoted with a tilde over the  $\sigma$ ) as

$$\tilde{\sigma}_{l+1}^0 = \begin{cases} J_{-\frac{l-1}{l+1}}^- J_{-\frac{l-3}{l+1}}^- \cdots J_{-\frac{1}{l+1}}^- \bar{J}_{-\frac{l-1}{l+1}}^- \bar{J}_{-\frac{l-3}{l+1}}^- \cdots \bar{J}_{-\frac{1}{l+1}}^- \sigma_{l+1} & (l+1) \text{ odd} \\ J_{-\frac{l-1}{l+1}}^- J_{-\frac{l-3}{l+1}}^- \cdots J_{-\frac{2}{l+1}}^- \bar{J}_{-\frac{l-1}{l+1}}^- \bar{J}_{-\frac{l-3}{l+1}}^- \cdots \bar{J}_{-\frac{2}{l+1}}^- (S_{l+1}^- \bar{S}_{l+1}^- \sigma_{l+1}) & (l+1) \text{ even.} \end{cases} \quad (3.21)$$

We can construct additional anti-chiral primaries just as in the case of chiral primaries. A chiral primary has a nonvanishing 2-point function with its corresponding anti-chiral primary. The chiral and anti-chiral twist operators are normalized such that the 2-point function is unity at unit separation. We use this fact later in the paper.

### 3.3.4 Ramond sector states

To get states in the Ramond sector we have to change the boundary conditions on fermions, making them periodic around the  $\sigma$  circle. This requires the insertion of a ‘spin field’. While this is not hard to do, we perform our computations by mapping the Ramond sector states to the NS sector by spectral flow. Thus we do not give the explicit structure of the Ramond sector states in this paper. In Appendix A we give a brief description of the Ramond vacua.

#### 4. The initial state, the final state, and the vertex operator

Our main computation addresses the following physical process. Consider a bound state of  $N_1$  D1 branes and  $N_5$  D5 branes, sitting at the origin of asymptotically flat space. As mentioned above, the CFT describing this bound state is in the Ramond sector, which has a number of degenerate ground states. We pick a particular Ramond ground state. Instead of describing this choice directly in the Ramond sector, we note that all Ramond sector ground states are obtained by one unit of spectral flow from chiral primary states of the NS sector. We pick the Ramond ground state that arises from the simplest chiral primary: the NS vacuum  $|\emptyset\rangle_{NS}$ . The gravity dual of this state can be described as follows [58, 59, 60, 19]: there is flat space at infinity, then a ‘neck’, then an  $AdS$  region, and then a ‘cap’, as pictured in Fig. 1(b). While the structure of the ‘cap’ depends on the choice of Ramond ground state, in the present case the structure is particularly simple: below  $r_b$  in Fig. 1(b) the geometry is a part of global  $AdS_3 \times S^3$ .

By itself such a D1D5 state is stable, and does not radiate energy. We therefore add an excitation to the D1D5 brane state. In the supergravity dual, the excitation we choose corresponds to adding a supergravity quantum sitting in the ‘cap’. The supergravity field we choose is a scalar  $\phi_{ij}$ , where  $i, j = 1, \dots, 4$  are vector indices valued in the  $T^4$  in (3.1). These scalars arise from the following fields:

- (i) A symmetric traceless matrix  $h_{ij}$  with  $i, j = 1, \dots, 4$  giving the transverse traceless gravitons with indices in the  $T^4$ .
- (ii) An antisymmetric matrix  $B_{ij}^{RR}$  giving the components of the Ramond-Ramond  $B$  field with indices in the torus.
- (iii) The dilaton, which is a scalar in the full 10-dimensional theory.

We can put all these scalars together into a  $4 \times 4$  matrix  $\phi_{ij}$ , with the symmetric traceless part coming from  $h_{ij}$ , the antisymmetric part from  $B_{ij}^{RR}$  and the trace from the dilaton. (Such a description was used for example in [4, 19]. But we may need to scale the above fields by some function. For instance, it is not the graviton,  $h_{ij}$ , but  $(H_5/H_1)^{\frac{1}{4}}h_{ij}$  which behaves as a minimal scalar in the 6-d space obtained by dimensional reduction on  $T^4$ .

The supergravity particle is described by its angular momenta in the  $S^3$  directions given by  $SU(2)_L \times SU(2)_R$  quantum numbers  $(\frac{l}{2}, m), (\frac{\bar{l}}{2}, \bar{m})$ ; and a ‘radial quantum number’,  $N$ , where  $N = 0$  gives the lowest energy state with the given angular momentum, and  $N = 1, 2, \dots$  give successively higher energy states.

Adding this quantum to the  $r < r_b$  region of the geometry corresponds to making an excitation of the D1D5 CFT. Since we compute all processes after spectral flowing to the NS sector, we should describe this excitation in the NS sector. In the NS sector, the excitation is a supersymmetry descendant of a chiral primary state, which is acted on  $N$  times with  $L_{-1}$  to further raise the energy. We describe the construction of this initial state in more detail below.

The process of interest is the emission of this supergravity particle from the cap out to infinity. The final state is thus simple: in the Ramond sector description we return to the Ramond ground state that we started with. In the spectral flowed NS sector description that we compute with, the final state is just the NS vacuum  $|\emptyset\rangle_{NS}$ .

The emission is caused by the interaction Lagrangian in Equation (2.52) which couples excitations in the CFT to modes at infinity; the general structure of this coupling was discussed in Section 2. We write down the vertex operator  $\hat{\mathcal{V}}$  which leads to the emission of the quanta  $\phi_{ij}$ , and compute the emission amplitude  $\langle f | \hat{\mathcal{V}} | i \rangle$ .

We now describe in detail the initial state, the final state, and the vertex operator.

#### 4.1 The initial state in the NS sector

Let us first write the state, and then explain its structure. The left and right parts of the state have similar forms, so we only write the left part (indicated by the subscript  $L$ ):

$$|\phi_{N+1}^{\frac{l}{2}, \frac{l}{2}-k}\rangle_L^{AA} = C_L L_{-1}^N (J_0^-)^k G_{-\frac{1}{2}}^{-A} \psi_{-\frac{1}{2}}^{+A} \sigma_{l+1}^0 |\emptyset\rangle_{NS}. \quad (4.1)$$

Let us describe the structure of this state starting with the elements on the rightmost end:

- (i) We start with the NS vacuum  $|\emptyset\rangle_{NS}$ . In this state each copy of the CFT is ‘singly wound’, and each copy is unexcited. In the supergravity dual, we have global AdS space with no particles in it.
- (ii) We apply the chiral primary  $\sigma_{l+1}^0$ , thereby twisting together  $l+1$  copies of the CFT into one ‘multiply wound’ component string. It also adds charge, so that we get a state with

$$h = m = \bar{h} = \bar{m} = \frac{l}{2}. \quad (4.2)$$

In the gravity dual, we have one supergravity quantum, with angular momenta  $(m, \bar{m})$ .

- (iii) We act with  $\psi_{-\frac{1}{2}}^{+A}$ . This increases both  $h$  and  $m$  by  $\frac{1}{2}$ , and so gives another chiral primary. We do the same for the right movers, so that overall the new state created is again bosonic. In the supergravity dual, it corresponds to a different bosonic quantum in the AdS.
- (iv) We act with elements of the ‘anomaly-free subalgebra’ of the chiral algebra:

- (a) The  $G_{-\frac{1}{2}}^{-A}$  changes the chiral primary to a supersymmetry descendant of the chiral primary, corresponding to a different supergravity particle in the gravity dual. Again, because we apply this supersymmetry operator on both left and right movers, the new supergravity quantum is bosonic. We now find that the indices carried by this quantum are those corresponding to a minimal scalar with both indices along the  $T^4$  in the gravity description. Thus we have finally arrived at the supergravity quantum that we wanted to consider.

- (b) The  $(J_0^-)^k$  rotate the quantum in the  $S^3$  directions. Before this rotation, the quantum numbers  $(m, \bar{m})$  were the highest allowed for the given supergravity particle state. The application of the  $(J_0^-)^k, (\bar{J}_0^-)^k$  give us other members of the  $SU(2)_L \times SU(2)_R$  multiplet.
- (c) The  $L_{-1}^N$  move and boost the quantum around in the  $AdS$ , thus increasing its energy and momentum.
- (v) Finally, we have a normalization constant. Below, we derive this in detail since the final expression for the radiation rate involves the factors appearing in this normalization.

#### 4.1.1 Normalizing the initial state

To find the normalization constant  $\mathcal{C}_L$ , we take the Hermitian conjugate to find

$${}_{AA}^L \langle \phi_{N+1}^{\frac{l}{2}, \frac{l}{2}-k} | = -\mathcal{C}_L^* {}_{NS} \langle \emptyset | \tilde{\sigma}_{l+1}^0 \epsilon_{\dot{A}\dot{B}} \psi_{\frac{1}{2}}^{-\dot{B}} \epsilon_{AB} G_{\frac{1}{2}}^{+B} (J_0^+)^k L_1^N, \quad (4.3)$$

and then compute the norm,

$$\begin{aligned} & {}_{AA}^L \langle \phi_{N+1}^{\frac{l}{2}, \frac{l}{2}-k} | \phi_{N+1}^{\frac{l}{2}, \frac{l}{2}-k} \rangle_L^{B\dot{B}} \\ &= -|\mathcal{C}_L|^2 \epsilon_{\dot{A}\dot{C}} \epsilon_{AC} {}_{NS} \langle \emptyset | \tilde{\sigma}_{l+1}^0 \psi_{\frac{1}{2}}^{-\dot{C}} G_{\frac{1}{2}}^{+C} (J_0^+)^k L_1^N L_{-1}^N (J_0^-)^k G_{-\frac{1}{2}}^{-B} \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 | \emptyset \rangle_{NS} \\ &= -|\mathcal{C}_L|^2 \epsilon_{\dot{A}\dot{C}} \epsilon_{AC} {}_{NS} \langle \emptyset | \tilde{\sigma}_{l+1}^0 \psi_{\frac{1}{2}}^{-\dot{C}} G_{\frac{1}{2}}^{+C} (J_0^+)^k \left( \prod_{j=1}^N j(2L_0 + j - 1) \right) (J_0^-)^k G_{-\frac{1}{2}}^{-B} \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 | \emptyset \rangle_{NS} \\ &= -\frac{N!(N+l+1)!}{(l+1)!} |\mathcal{C}_L|^2 \epsilon_{\dot{A}\dot{C}} \epsilon_{AC} {}_{NS} \langle \emptyset | \tilde{\sigma}_{l+1}^0 \psi_{\frac{1}{2}}^{-\dot{C}} G_{\frac{1}{2}}^{+C} (J_0^+)^k (J_0^-)^k G_{-\frac{1}{2}}^{-B} \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 | \emptyset \rangle_{NS} \\ &= -\frac{k! l!}{(l-k)!} \frac{N!(N+l+1)!}{(l+1)!} |\mathcal{C}_L|^2 \epsilon_{\dot{A}\dot{C}} \epsilon_{AC} {}_{NS} \langle \emptyset | \tilde{\sigma}_{l+1}^0 \psi_{\frac{1}{2}}^{-\dot{C}} G_{\frac{1}{2}}^{+C} G_{-\frac{1}{2}}^{-B} \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 | \emptyset \rangle_{NS} \\ &= -2\delta_A^B \frac{k! l!}{(l-k)!} \frac{N!(N+l+1)!}{(l+1)!} |\mathcal{C}_L|^2 \epsilon_{\dot{A}\dot{C}} {}_{NS} \langle \emptyset | \tilde{\sigma}_{l+1}^0 \psi_{\frac{1}{2}}^{-\dot{C}} (L_0 + J_0^3) \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 | \emptyset \rangle_{NS} \quad (4.4) \end{aligned}$$

Proceeding with the calculation, one finds

$$\begin{aligned} & {}_{AA}^L \langle \phi_{N+1}^{\frac{l}{2}, \frac{l}{2}-k} | \phi_{N+1}^{\frac{l}{2}, \frac{l}{2}-k} \rangle_L^{B\dot{B}} = -2(l+1) \delta_A^B \frac{k! l!}{(l-k)!} \frac{N!(N+l+1)!}{(l+1)!} |\mathcal{C}_L|^2 \epsilon_{\dot{A}\dot{C}} {}_{NS} \langle \emptyset | \tilde{\sigma}_{l+1}^0 \psi_{\frac{1}{2}}^{-\dot{C}} \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 | \emptyset \rangle_{NS} \\ &= -2\delta_A^B \frac{N!(N+l+1)! k!}{(l-k)!} |\mathcal{C}_L|^2 \epsilon_{\dot{A}\dot{C}} {}_{NS} \langle \emptyset | \tilde{\sigma}_{l+1}^0 \psi_{\frac{1}{2}}^{-\dot{C}} \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 | \emptyset \rangle_{NS} \\ &= 2\delta_A^B \delta_{\dot{A}}^{\dot{B}} \frac{N!(N+l+1)! k!}{(l-k)!} (l+1) |\mathcal{C}_L|^2, \quad (4.5) \end{aligned}$$

where we have used the fact that the chiral primary twist operators are correctly normalized. The factor of  $l+1$  comes from the fermion anticommutator, since in the twisted sector there

are  $l + 1$  copies of the fermion field that go into what we call  $\psi$ . One can understand this factor most easily by using Equation (A.24). By demanding that

$${}^L_{A\dot{A}}\langle\phi^{\frac{l}{2},\frac{l}{2}-k}_{N+1}|\phi^{\frac{l}{2},\frac{l}{2}-k}_{N+1}\rangle_L^{B\dot{B}} = \delta_A^B\delta_{\dot{A}}^{\dot{B}}, \quad (4.6)$$

we conclude that the normalized state (with the left *and* right parts) is

$$\begin{aligned} |\phi\rangle^{A\dot{A}B\dot{B}} &= \sqrt{\frac{(l-k)!(l-\bar{k})!}{4N!\bar{N}!(N+l+1)!(\bar{N}+l+1)!k!\bar{k}!(l+1)^2}} \\ &\times L_{-1}^N (J_0^-)^k G_{-\frac{1}{2}}^{-A}\psi_{-\frac{1}{2}}^{+\dot{A}} \bar{L}_{-1}^{\bar{N}} (\bar{J}_0^-)^{\bar{k}} \bar{G}_{-\frac{1}{2}}^{-\dot{B}}\bar{\psi}_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 |\emptyset\rangle_{NS}. \end{aligned} \quad (4.7)$$

In this computation we use the identity

$$\prod_{j=1}^k (jl - j(j-1)) = \frac{k!l!}{(l-k)!}. \quad (4.8)$$

#### 4.1.2 The state in $SO(4)_I$ notation

In the gravity description it is natural to write the quantum as  $\phi_{ij}$ , with vector indices  $ij$  of the internal symmetry group  $SO(4)_I$  of the  $T^4$  directions. For CFT computations it is more useful to use indices  $A\dot{A}$  for  $SU(2)_1 \times SU(2)_2$ , as we do above. The conversion is achieved by

$$|\phi^{\frac{l}{2},\frac{l}{2}-k}_{N+1}\rangle_L^i = \frac{1}{\sqrt{2}}(\sigma^i)^{A\dot{A}}\epsilon_{AB}\epsilon_{\dot{A}\dot{B}}|\phi^{\frac{l}{2},\frac{l}{2}-k}_{N+1}\rangle_L^{B\dot{B}}. \quad (4.9)$$

We then have

$${}_L^i\langle\phi^{\frac{l}{2},\frac{l}{2}-k}_{N+1}|\phi^{\frac{l}{2},\frac{l}{2}-k}_{N+1}\rangle_L^j = \delta^{ij}. \quad (4.10)$$

Similarly, one typically labels the angular momentum eigenstates in terms of  $(l, m_\psi, m_\phi)$ , instead of  $(l, m, \bar{m})$ . The two bases are related via

$$\begin{aligned} m_\psi &= -(m + \bar{m}) \\ m_\phi &= m - \bar{m}, \end{aligned} \quad (4.11)$$

where the values on the right-hand side are the angular momenta of the initial state in the NS sector.

#### 4.2 The final state

In the supergravity description the initial state had one quantum in it. The emission process of interest leads to the emission of this quantum. Thus the final state has no quanta, and in the NS description is just the vacuum

$$|f\rangle = |\emptyset\rangle_{NS}. \quad (4.12)$$

### 4.3 The vertex operator

We need the vertex operator that emits the supergravity quantum described by the initial state  $|i\rangle$ . Vertex operators describing supergravity particles are given by chiral primaries and their descendants under the anomaly-free part of the chiral algebra.

For the process of interest the emission vertex must have appropriate charges to couple to the supergravity field under consideration. Thus, one naturally concludes that the vertex operator has essentially the same structure as the state  $|i\rangle$ , with two differences. First, the operator has charges that are opposite to the charges carried by the state. (We get a nonvanishing inner product between  $|i\rangle$  and the Hermitian *conjugate* of  $|i\rangle$ .) Second, the operator does not have the  $L_{-1}$  modes present in the description of the CFT state. This is because applying an  $L_{-1}$  mode is equivalent to translating the location of the vertex insertion, and we have already chosen the insertion to be the point  $(\sigma, \tau)$ . Note that after applying the supercurrent to give the operator the correct  $SO(4)_I$  index structure, one finds that the operator already has the correct weight to couple to a minimal scalar in Equation (2.52) and form a scale invariant action.

The vertex operator, then, is given by (we drop the hat on the vertex operator from now on)

$$\tilde{\mathcal{V}}_{l,l-k-\bar{k},k-\bar{k}}^{A\dot{A}B\dot{B}}(\sigma, \tau) = \frac{1}{2} \sqrt{\frac{(l-k)!(l-\bar{k})!}{(l+1)^2(l+1)!^2 k! \bar{k}!}} (J_0^+)^k (\bar{J}_0^+)^{\bar{k}} G_{-\frac{1}{2}}^{+A} \psi_{-\frac{1}{2}}^{-\dot{A}} \bar{G}_{-\frac{1}{2}}^{+B} \bar{\psi}_{-\frac{1}{2}}^{-\dot{B}} \tilde{\sigma}_{l+1}^0(\sigma, \tau). \quad (4.13)$$

The subscript on the vertex operator  $\mathcal{V}_{l,m_\psi,m_\phi}$  are the  $SO(4)_E$  angular momenta labels. Again the normalization is a crucial part of the final amplitude, so we perform it in more detail below.

Note that for  $l = 0$ , the vertex operator reduces to  $[\partial X]^{A\dot{A}}[\bar{\partial} X]^{B\dot{B}}$ , the old ‘effective string’ coupling found by expanding the DBI action [4].

#### 4.3.1 Mapping to the complex plane

Before normalizing the vertex operator, we first map the operator from the cylinder onto the complex plane via  $z = e^{\tau + i\sigma}$ . The vertex operator has weight  $\frac{l}{2} + 1$  on both the left and the right, so we get

$$\begin{aligned} \tilde{\mathcal{V}}_{l,l-k-\bar{k},k-\bar{k}}^{A\dot{A}B\dot{B}}(\sigma, \tau) &= |z|^{l+2} \frac{1}{2} \sqrt{\frac{(l-k)!(l-\bar{k})!}{(l+1)^2(l+1)!^2 k! \bar{k}!}} \left( (J_0^+)^k (\bar{J}_0^+)^{\bar{k}} G_{-\frac{1}{2}}^{+A} \psi_{-\frac{1}{2}}^{-\dot{A}} \bar{G}_{-\frac{1}{2}}^{+B} \bar{\psi}_{-\frac{1}{2}}^{-\dot{B}} \tilde{\sigma}_{l+1}^0(z, \bar{z}) \right)_{z, \bar{z}} \\ &= |z|^{l+2} \mathcal{V}_{l,l-k-\bar{k},k-\bar{k}}^{A\dot{A}B\dot{B}}(z, \bar{z}). \end{aligned} \quad (4.14)$$

#### 4.3.2 Normalizing the vertex operator

The left part of the vertex operator is given by

$$\mathcal{V}_{L;l,k}^{A\dot{A}}(z) = N_L \left( (J_0^+)^k G_{-\frac{1}{2}}^{+A} \psi_{-\frac{1}{2}}^{-\dot{A}} \tilde{\sigma}_{l+1}^0(z) \right)_z. \quad (4.15)$$

We need to normalize the vertex operator. To that end, we begin by writing its Hermitian conjugate:

$$\mathcal{V}_{L;l,k}^{A\dot{A}}{}^\dagger(z) = -(-1)^{k+1} \epsilon_{AB} \epsilon_{\dot{A}\dot{B}} N_L^* \left( (J_0^-)^k G_{-\frac{1}{2}}^{-B} \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0(z) \right)_z = \epsilon_{AB} \epsilon_{\dot{A}\dot{B}} \mathcal{V}_{l,2l-k}^{B\dot{B}}(z), \quad (4.16)$$

where the second equality is the condition needed to ensure the total interaction action is Hermitian.

The factor of  $(-1)^{k+1}$  comes from the  $G$  and the  $J_0$ 's. We illustrate below with  $J_0^+$ :

$$\begin{aligned} [(J_0^+)_z]^\dagger &= \left[ \oint_z \frac{dz'}{2\pi i} J^+(z') \right]^\dagger \\ &= - \oint_{\bar{z}} \frac{d\bar{z}'}{2\pi i} J^-\left(\frac{1}{\bar{z}'}\right) \frac{1}{\bar{z}'^2} \\ &= - \oint_{\frac{1}{\bar{z}}} \frac{d\xi}{2\pi i} J^-(\xi) \\ &= -(J_0^-)_{\frac{1}{\bar{z}}}, \end{aligned} \quad (4.17)$$

where when making the change of variables  $\xi = 1/\bar{z}'$  there are two minus signs. One comes from the Jacobian  $d\bar{z}' = -1/\xi^2 d\xi$ , and the other comes from making the contour counter-clockwise. The  $G_{-\frac{1}{2}}$  behaves in the same way; however, the  $\psi_{-\frac{1}{2}}$  is different:

$$\begin{aligned} \left[ (\psi_{-\frac{1}{2}}^{-\dot{A}})_z \right]^\dagger &= \left[ \oint_z \frac{dz'}{2\pi i} \frac{\psi^{-\dot{A}}(z')}{z' - z} \right]^\dagger \\ &= -(-\epsilon_{-+} \epsilon_{\dot{A}\dot{B}}) \oint_{\bar{z}} \frac{d\bar{z}'}{2\pi i} \psi^{+\dot{B}}\left(\frac{1}{\bar{z}'}\right) \frac{1}{\bar{z}'(\bar{z}' - \bar{z})} \\ &= -\epsilon_{\dot{A}\dot{B}} \oint_{\frac{1}{\bar{z}}} \frac{d\xi}{2\pi i} \psi^{+\dot{B}}(\xi) \frac{1}{\xi \left(\frac{1}{\xi} - \bar{z}\right)} \\ &= \frac{\epsilon_{\dot{A}\dot{B}}}{\bar{z}} \oint_{\frac{1}{\bar{z}}} \frac{d\xi}{2\pi i} \frac{\psi^{+\dot{B}}(\xi)}{\xi - \frac{1}{\bar{z}}} \\ &= \frac{\epsilon_{\dot{A}\dot{B}}}{\bar{z}} (\psi_{-\frac{1}{2}}^{+\dot{B}})_{\frac{1}{\bar{z}}}; \end{aligned} \quad (4.18)$$

it receives an extra minus sign from the integrand.

We use the notation

$$\langle \cdot \rangle = {}_{NS} \langle \emptyset | \cdot | \emptyset \rangle_{NS} \quad (4.19)$$

for the NS-vacuum expectation value. Proceeding with the normalization, the 2-point function



is given by

$$\begin{aligned}
\left\langle \mathcal{V}_{L;l,k}^{A\dot{A}\dagger}(z) \mathcal{V}_{L;l,k}^{B\dot{B}}(0) \right\rangle &= (-1)^{k+2} |N_L|^2 \epsilon_{AC} \epsilon_{\dot{A}\dot{C}} \left\langle \left( (J_0^-)^k G_{-\frac{1}{2}}^{-C} \psi_{-\frac{1}{2}}^{+\dot{C}} \sigma_{l+1}^0(z) \right)_z \left( (J_0^+)^k G_{-\frac{1}{2}}^{+B} \psi_{-\frac{1}{2}}^{-\dot{B}} \tilde{\sigma}_{l+1}^0(0) \right)_0 \right\rangle \\
&= -|N_L|^2 \epsilon_{AC} \epsilon_{\dot{A}\dot{C}} \left\langle \left( \psi_{-\frac{1}{2}}^{+\dot{C}} \sigma_{l+1}^0(z) \right)_z \left( G_{-\frac{1}{2}}^{-C} (J_0^-)^k (J_0^+)^k G_{-\frac{1}{2}}^{+B} \psi_{-\frac{1}{2}}^{-\dot{B}} \tilde{\sigma}_{l+1}^0(0) \right)_0 \right\rangle \\
&= -|N_L|^2 \epsilon_{AC} \epsilon_{\dot{A}\dot{C}} \frac{k! l!}{(l-k)!} \left\langle \left( \psi_{-\frac{1}{2}}^{+\dot{C}} \sigma_{l+1}^0(z) \right)_z \left( G_{-\frac{1}{2}}^{-C} G_{-\frac{1}{2}}^{+B} \psi_{-\frac{1}{2}}^{-\dot{B}} \tilde{\sigma}_{l+1}^0(0) \right)_0 \right\rangle \\
&= 2|N_L|^2 \delta_A^B \epsilon_{\dot{A}\dot{C}} \frac{k! l!}{(l-k)!} \left\langle \left( \psi_{-\frac{1}{2}}^{+\dot{C}} \sigma_{l+1}^0(z) \right)_z \left( L_{-1} \psi_{-\frac{1}{2}}^{-\dot{B}} \tilde{\sigma}_{l+1}^0(0) \right)_0 \right\rangle \\
&= 2|N_L|^2 \delta_A^B \epsilon_{\dot{A}\dot{C}} \frac{k! l!}{(l-k)!} \lim_{v \rightarrow 0} \partial_v \left\langle \left( \psi_{-\frac{1}{2}}^{+\dot{C}} \sigma_{l+1}^0(z) \right)_z \left( \psi_{-\frac{1}{2}}^{-\dot{B}} \tilde{\sigma}_{l+1}^0(v) \right)_v \right\rangle \\
&= 2|N_L|^2 \delta_A^B \epsilon_{\dot{A}\dot{C}} \epsilon^{\dot{C}\dot{B}} \frac{k! l!}{(l-k)!} \lim_{v \rightarrow 0} \partial_v \frac{l+1}{(z-v)^{l+1}} \\
&= 2|N_L|^2 \delta_A^B \delta_{\dot{A}}^{\dot{B}} \frac{k! (l+1)!}{(l-k)!} \frac{1}{z^{l+2}}. \tag{4.20}
\end{aligned}$$

The factor of  $l+1$  has the same origin as in the normalization of the initial state. Using the above, one finds

$$N_L = \frac{1}{\sqrt{2}} \sqrt{\frac{(l-k)!}{k! (l+1)! (l+1)}}, \tag{4.21}$$

and thus the left part of the vertex operator is

$$\mathcal{V}_{L;l,k}^{A\dot{A}}(z) = \frac{1}{\sqrt{2}} \sqrt{\frac{(l-k)!}{k! (l+1)! (l+1)}} \left( (J_0^+)^k G_{-\frac{1}{2}}^{+A} \psi_{-\frac{1}{2}}^{-\dot{A}} \tilde{\sigma}_{l+1}^0(z) \right)_z. \tag{4.22}$$

The normalization is chosen such that the vertex operator in the complex plane satisfies

$$\begin{aligned}
\langle \mathcal{V}_{l,-m_\psi,-m_\phi}^{A\dot{A}B\dot{B}}(z) \mathcal{V}_{l,m_\psi,m_\phi}^{C\dot{C}D\dot{D}}(0) \rangle &= \frac{\epsilon^{AC} \epsilon^{\dot{A}\dot{C}} \epsilon^{BD} \epsilon^{\dot{B}\dot{D}}}{|z|^{l+2}} \\
\langle \mathcal{V}_{l,-m_\psi,-m_\phi}^{ij}(z) \mathcal{V}_{l,m_\psi,m_\phi}^{kl}(0) \rangle &= \frac{\delta^{ik} \delta^{jl}}{|z|^{l+2}}. \tag{4.23}
\end{aligned}$$

Note that this is the normalization of the operator corresponding to one particular way of permuting  $l+1$  copies of the CFT. As mentioned earlier, the actual vertex operator coupling to  $\phi_{ij}$  is a symmetrized sum over all possible ways of permtuing  $l+1$  copies from the  $N_1 N_5$  available copies. We discuss the combinatorics of this choice in Section 7 below, and at that time note the extra normalization factor which is needed to agree with (2.27).

## 5. Using spectral flow

We wish to relate a CFT amplitude computed in the NS sector,

$$\mathcal{A}' = \langle f' | \mathcal{V}(z, \bar{z}) | i' \rangle, \tag{5.1}$$

to an amplitude in the Ramond sector, since the physical D1D5 system has its fermions periodic around the  $y$  circle. In this section, we show how to spectral flow [61, 62, 63] the computation in the NS sector to the physical problem in the R sector. Furthermore, we find that we can relate this NS sector computation to a whole family of Ramond sector amplitudes, and each member of the family describes a different physical emission process.

If spectral flowing the states  $|i'\rangle$  and  $|f'\rangle$  by  $\alpha$  units is given by

$$|i'\rangle \mapsto |i\rangle = \mathcal{U}_\alpha |i'\rangle \quad \langle f'| \mapsto \langle f| = \langle f'| \mathcal{U}_{-\alpha}, \quad (5.2)$$

then we can compute the amplitude in the Ramond sector by using

$$\mathcal{A}_{\text{Ramond}} = \langle f| \mathcal{V}(z, \bar{z}) |i\rangle = (\langle f| \mathcal{U}_\alpha) (\mathcal{U}_{-\alpha} \mathcal{V} \mathcal{U}_\alpha) (\mathcal{U}_{-\alpha} |i\rangle) = \langle f'| \mathcal{V}'(z, \bar{z}) |i'\rangle. \quad (5.3)$$

Note that one finds  $\mathcal{V}'$  by spectral flowing  $\mathcal{V}$  by  $-\alpha$  units.

We need to determine how the vertex operator transforms under spectral flow. First, we demonstrate that the  $G\psi$  part is unaffected, since

$$\begin{aligned} \left( G_{-\frac{1}{2}}^{+A} \psi_{-\frac{1}{2}}^{-\dot{A}} \right)_z &= \oint_z \frac{dz_1}{2\pi i} \oint_z \frac{dz_2}{2\pi i} \frac{G^{+A}(z_1) \psi^{-\dot{A}}(z_2)}{z_2 - z} \\ &= - \oint_z \frac{dz_1}{2\pi i} \oint_z \frac{dz_2}{2\pi i} \frac{[\partial X(z_2)]^{\dot{A}A}}{(z_2 - z)(z_1 - z_2)} \end{aligned} \quad (5.4)$$

and the bosons are unaffected by spectral flow.

Therefore, we need only spectral flow the  $k$   $J_0^+$ 's and the chiral primary. The effect of spectral flow by *negative*  $\alpha$  units on chiral ( $h = m$ ) and anti-chiral primaries ( $h = -m$ ) is very simple:

$$\mathcal{O}'_{\text{c.p.}}(z) = z^{\alpha m} \mathcal{O}_{\text{c.p.}}(z) \quad \mathcal{O}'_{\text{a.c.p.}}(z) = z^{\alpha m} \mathcal{O}_{\text{a.c.p.}}(z). \quad (5.5)$$

One can see this most directly after bosonizing the fermions; see Appendix A.9 for details.

Under spectral flow by  $-\alpha$  units the  $J^\pm$  transform as

$$J^\pm(z) \mapsto z^{\pm\alpha} J^\pm(z), \quad (5.6)$$

from which we see that

$$(J_0^+)_z = \oint_z \frac{dz'}{2\pi i} J^+(z') \mapsto \oint_z \frac{dz'}{2\pi i} J^+(z') z'^\alpha = z^\alpha (J_0^+)_z + \alpha z^{\alpha-1} (J_1^+)_z + \dots \quad (5.7)$$

Only the first term contributes since the positive modes annihilate a chiral primary. Therefore, we conclude that spectral flowing the vertex operator by  $-\alpha$  units has the effect of

$$\mathcal{V}'_{l,l-k-\bar{k},k-\bar{k}}(z, \bar{z}) = z^{-\alpha(\frac{l}{2}-k)} \bar{z}^{-\alpha(\frac{l}{2}-\bar{k})} \mathcal{V}_{l,l-k-\bar{k},k-\bar{k}}(z, \bar{z}). \quad (5.8)$$

Thus we observe that we can spectral flow the initial and final states, keep the vertex operator unchanged, and compute the amplitude

$$\mathcal{A}' = \langle f'| \mathcal{V}(z, \bar{z}) |i'\rangle. \quad (5.9)$$

The result we want,  $\mathcal{A}_{\text{Ramond}}$ , is then given by

$$\mathcal{A}_{\text{Ramond}} = z^{-\alpha(\frac{l}{2}-k)} \bar{z}^{-\bar{\alpha}(\frac{l}{2}-\bar{k})} \langle f' | \mathcal{V}(z, \bar{z}) | i' \rangle = z^{-\alpha(\frac{l}{2}-k)} \bar{z}^{-\bar{\alpha}(\frac{l}{2}-\bar{k})} \mathcal{A}'. \quad (5.10)$$

Here  $\alpha$  is chosen to have a value that spectral flows from the NS to the Ramond sector, but this can be achieved by *any* odd integral value of  $\alpha$ :

$$\alpha = (2n + 1) \quad \bar{\alpha} = (2\bar{n} + 1) \quad n, \bar{n} \in \mathbb{Z}. \quad (5.11)$$

For these values of  $\alpha$  the initial and final states have weight and charge

$$\begin{aligned} h &= h' + (2n + 1)m' + (2n + 1)^2 \frac{c_{\text{tot.}}}{24} \\ m &= m' + (2n + 1) \frac{c_{\text{tot.}}}{12}, \end{aligned} \quad (5.12)$$

where  $c_{\text{tot.}}$  is  $c = 6$  times the number of copies being spectral flowed. A similar relation holds on the right sector.

In our present computation in the NS sector, we have

$$\begin{aligned} h'_i &= \frac{l}{2} + N + 1 & h'_f &= 0 \\ m'_i &= \frac{l}{2} - k & m'_f &= 0. \end{aligned} \quad (5.13)$$

In the next section we look at the Ramond sector process for  $\alpha = \bar{\alpha} = 1$ . In this case the weights and charges of the Ramond sector states are

$$\begin{aligned} h_i &= \frac{l}{2} + N + 1 + \left( \frac{l}{2} - k \right) + \frac{l+1}{4} & h_f &= (l+1) \frac{1}{4} \\ m_i &= \frac{l}{2} - k + \frac{l+1}{2} & m_f &= (l+1) \frac{1}{2}. \end{aligned} \quad (5.14)$$

We see that the final state has the weight and charge of the ‘spin-up’ Ramond vacuum, while the initial state has the correct weight and charge above the Ramond vacuum. Although the current calculation is  $\alpha = 1$ , we leave  $\alpha$  as an explicit parameter in following calculations for later use and illustration.

In section 9, the full  $\alpha$  and  $\bar{\alpha}$  dependence is of physical interest, since how big  $\alpha$  and  $\bar{\alpha}$  are roughly corresponds to how nonextremal the initial state is.

## 6. Evaluating the CFT amplitude

Let us now compute the amplitude

$$\mathcal{A}'^{A\dot{A}}(\sigma, \tau) = \langle f' | \tilde{\mathcal{V}}(\sigma, \tau) | i' \rangle = |z|^{l+2} \langle f' | \mathcal{V}(z, \bar{z}) | i' \rangle. \quad (6.1)$$

We choose the charges of the initial state and the vertex operator so that we get a nonvanishing amplitude. The nonvanishing amplitude is

$$\begin{aligned} \mathcal{A}'_L{}^{A\dot{A}} &= \frac{1}{\sqrt{2}}(\sigma^{\bar{i}})_{B\dot{B}} z^{\frac{l}{2}+1} \sqrt{\frac{(l-k)!}{2(l+1)(l+1)!k!}} {}_{NS}\langle \emptyset | \left( (J_0^+)^k G_{-\frac{1}{2}}^{+A} \psi_{-\frac{1}{2}}^{-\dot{A}} \tilde{\sigma}_{l+1}^0 \right)_z \\ &\quad \times \sqrt{\frac{(l-k)!}{2N!(N+l+1)!k!(l+1)}} \left( (J_0^-)^k G_{-\frac{1}{2}}^{-B} L_{-1}^N \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 \right)_0 |\emptyset \rangle_{NS}, \end{aligned} \quad (6.2)$$

where  $A, \dot{A}$  and  $\bar{i}$  are free indices. The  $\bar{i}$  is the index of the initial state excitation and  $A, \dot{A}$  are the indices on the vertex operator. We have commuted the  $L_{-1}$ 's to the right for calculational convenience.

To evaluate the correlator, we first note the subscript  $z$  on the first parenthetical expression indicates that the contours for the modes circle  $z$ . Since there are no obstructions, we can shift the contour to orbit the origin instead. We have

$$\begin{aligned} (-1)^{k+1} \left\langle \left( \psi_{-\frac{1}{2}}^{-\dot{A}} \tilde{\sigma}_{l+1}^0 \right)_z \left( G_{-\frac{1}{2}}^{+A} (J_0^+)^k (J_0^-)^k G_{-\frac{1}{2}}^{-B} L_{-1}^N \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 \right)_0 \right\rangle \\ = (-1)^{k+1} 2 \frac{k!l!}{(l-k)!} \epsilon^{AB} \epsilon^{+-} \left\langle \left( \psi_{-\frac{1}{2}}^{-\dot{A}} \tilde{\sigma}_{l+1}^0 \right)_z \left( L_{-1}^{N+1} \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 \right)_0 \right\rangle. \end{aligned} \quad (6.3)$$

For the final step, we should note the action of  $L_{-1}$  on a primary field  $\mathcal{O}$  is

$$L_{-1}\mathcal{O}(0) = \oint \frac{dz}{2\pi i} T(z)\mathcal{O}(0) = \partial\mathcal{O}(0); \quad (6.4)$$

therefore, we may write

$$\begin{aligned} \left\langle \left( \psi_{-\frac{1}{2}}^{-\dot{A}} \tilde{\sigma}_{l+1}^0 \right)_z \left( L_{-1}^{N+1} \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 \right)_0 \right\rangle &= \lim_{v \rightarrow 0} \partial_v^{N+1} \left\langle \left( \psi_{-\frac{1}{2}}^{-\dot{A}} \tilde{\sigma}_{l+1}^0 \right)_z \left( \psi_{-\frac{1}{2}}^{+\dot{B}} \sigma_{l+1}^0 \right)_v \right\rangle \\ &= -\epsilon^{\dot{A}\dot{B}} \lim_{v \rightarrow 0} \partial_v^{N+1} \frac{l+1}{(z-v)^{l+1}} \\ &= -\epsilon^{\dot{A}\dot{B}} \frac{(N+l+1)!(l+1)}{l!} \frac{1}{z^{l+N+2}}. \end{aligned} \quad (6.5)$$

Finally, one finds the left amplitude reduces to the simple form

$$\mathcal{A}'_L{}^{A\dot{A}} = (-1)^k \frac{1}{\sqrt{2}} (\sigma^{\bar{i}})^{A\dot{A}} \frac{1}{z^{\frac{l}{2}+N+1}} \sqrt{\binom{N+l+1}{N}}. \quad (6.6)$$

From Equation (5.10), we find that the left part of the CFT amplitude in the Ramond sector is given by

$$\begin{aligned} \mathcal{A}_L^{A\dot{A}} &= z^{-\alpha(\frac{l}{2}-k)} \mathcal{A}'_L{}^{A\dot{A}} \\ &= (-1)^k \frac{1}{\sqrt{2}} (\sigma^{\bar{i}})^{A\dot{A}} \frac{1}{z^{(1+\alpha)\frac{l}{2}-\alpha k+N+1}} \sqrt{\binom{N+l+1}{N}}. \end{aligned} \quad (6.7)$$

Finally, converting back to  $SO(4)$  indices for the vertex operator, one gets in the Ramond sector

$$\begin{aligned}\mathcal{A}_L^{\bar{i}}(z) &= \frac{1}{\sqrt{2}}(\sigma^{\bar{i}})_{A\dot{A}}\mathcal{A}_L^{A\dot{A}} \\ &= (-1)^{k+1} \frac{1}{z^{(1+\alpha)\frac{l}{2}-\alpha k+N+1}} \sqrt{\binom{N+l+1}{N}},\end{aligned}\tag{6.8}$$

The free index  $\bar{i}$  and a similar index from the right movers  $\bar{j}$  correspond to the indices  $\phi_{ij}$  for the field coupling to the emission vertex.

## 7. Combinatorics

The full CFT has  $N_1 N_5$  copies of the basic  $c = 6$  CFT. In the above section, we took a set of  $l + 1$  copies twisted together, and look at an emission process where the emission vertex untwists these copies. Now, we must put this computation in its full CFT context, by doing the following:

- (i) We must compute the combinatorics of how we pick the particular way of twisting  $l + 1$  copies in the initial state from all  $N_1 N_5$  copies.
- (ii) We must similarly consider all the ways that the vertex operator can twist copies. This allows us to normalize the vertex operator in the full theory so that we reproduce (2.27).
- (iii) We can take the limit  $N_1 N_5 \rightarrow \infty$  to get the ‘classical limit’ of the D1D5 system; the result in this limit should agree with the computation in the dual supergravity theory.

In fact we start with something a little more general. We assume that the initial state has  $\nu$  quanta of the same kind, and let the emission process lead to the final state with  $\nu - 1$  quanta. We then observe a Bose enhancement of the emission amplitude by a factor  $\sqrt{\nu}$ , which agrees with the enhancement observed in both CFT and dual gravity computations in [14].

### 7.1 The initial state

We wish to have  $\nu$  excitations, each of which involve twisting together  $l + 1$  copies of the  $c = 6$  CFT. We can pick the needed copies in any way from the full set of  $N_1 N_5$  copies, and because of the orbifold symmetry between these copies the state must be a symmetrized sum over these possibilities:

$$|\Psi_\nu\rangle = \mathcal{C}_\nu \left[ |\psi_\nu^1\rangle + |\psi_\nu^2\rangle + \dots \right],\tag{7.1}$$

where  $\mathcal{C}_\nu$  is the overall normalization and each  $|\psi_\nu^i\rangle$  is individually normalized. To understand what we are doing better, note that the state  $|\psi_\nu^1\rangle$  can be written schematically as

$$|\psi_\nu^1\rangle = |[12 \cdots (l+1)][(l+2) \cdots 2(l+1)] \cdots [(\nu(l+1) - l) \cdots \nu(l+1)]\rangle,\tag{7.2}$$

where the numbers in the square brackets are indicating particular ways of twisting individual strands corresponding to particular cycles of the permutation group. For instance,

$$|[1234]\rangle, \quad (7.3)$$

indicates that we twist strand 1 into strand 2 into strand 3 into strand 4 into strand 1 and leave strands 5 through  $N_1 N_5$  untwisted.

Our first task is to determine the number of terms in Equation (7.1) and thereby its normalization  $\mathcal{C}_\nu$ . To count the number of states we imagine constructing one of these states and see how many choices we have along the way. First, we choose  $\nu(l+1)$  of the total  $N_1 N_5$  strands that are going to be twisted in some way. The remaining strands are untwisted. Those  $\nu(l+1)$  strands must now be broken into sets of  $l+1$ . To do this, we first choose  $l+1$  of the  $\nu(l+1)$ , then the next set of  $l+1$  from the remaining  $(\nu-1)(l+1)$ , and so on. Note that  $|[12][34]\rangle = |[34][12]\rangle$ , and so there is no sense in talking about the ‘first’ set versus the ‘second set’. Therefore we should divide by the number of ways to rearrange the  $\nu$  sets of  $l+1$ . Finally, we should choose a particular cycle for each set of  $(l+1)$ ; since it does not matter where we start on the final cycle, this gives a factor  $l!$  for each twisted cycle. Putting all of these factors together yields the number of terms in Equation (7.1),

$$\begin{aligned} N_{\text{terms}} &= \binom{N_1 N_5}{\nu(l+1)} \times \binom{\nu(l+1)}{l+1} \binom{(\nu-1)(l+1)}{l+1} \dots \binom{l+1}{l+1} \times \frac{1}{\nu!} \times (l!)^\nu \\ &= \frac{(N_1 N_5)!}{(l+1)^\nu \nu! [N_1 N_5 - \nu(l+1)]!}. \end{aligned} \quad (7.4)$$

Without loss of generality, let us choose  $\mathcal{C}_\nu$  to be real, which gives

$$\mathcal{C}_\nu = \left[ \frac{(N_1 N_5)!}{(l+1)^\nu \nu! [N_1 N_5 - \nu(l+1)]!} \right]^{-\frac{1}{2}}. \quad (7.5)$$

## 7.2 The final state

The final state is simply  $|\Psi_{\nu-1}\rangle$ , with its corresponding normalization  $\mathcal{C}_{\nu-1}$ .

## 7.3 The vertex operator

The vertex operator can twist together any  $l+1$  copies of the CFT with any  $l+1$ -cycle, and it should be written as a symmetrized sum over these possibilities:

$$\mathcal{V}_{\text{sym}} = \mathcal{C} \sum_i \mathcal{V}_i. \quad (7.6)$$

Since the joined copies form a single long loop, the order of copies matters but not which copy is the ‘first one’ in the loop. Thus the number of terms in the sum is

$$\binom{N_1 N_5}{l+1} l! = \frac{(N_1 N_5)!}{(l+1) [N_1 N_5 - (l+1)]!}. \quad (7.7)$$

This gives the normalization

$$\mathcal{C} = \left[ \frac{(N_1 N_5)!}{(l+1) [N_1 N_5 - (l+1)]!} \right]^{-\frac{1}{2}}. \quad (7.8)$$

## 7.4 The amplitude

To compute the amplitude

$$\langle \Psi_{\nu-1} | \mathcal{V}_{\text{sym}} | \Psi_{\nu} \rangle, \quad (7.9)$$

we have to count the different ways that terms in the initial state can combine with terms in the vertex operator and terms in the final state to produce a nonzero amplitude. For a given initial state term  $|\psi_{\nu}^i\rangle$ , there are exactly  $\nu$  vertex operators  $\mathcal{V}_i$  that can de-excite it into a final state. There is only one final state that works, obviously. Thus the number of ways that we can get a nonzero amplitude is simply

$$\nu N_{\text{terms}} = \frac{\nu}{\mathcal{C}_{\nu}^2}.$$

Let

$$\langle \psi_{\nu-1}^1 | \mathcal{V}_1 | \psi_{\nu}^1 \rangle \quad (7.10)$$

be the amplitude obtained by using only one allowed initial state  $|\psi_{\nu}^1\rangle$  from the set in Equation (7.1) and one allowed vertex operator  $\mathcal{V}_1$  from the set in Equation (7.6). Then we have

$$\begin{aligned} \langle \Psi_{\nu-1} | \mathcal{V}_{\text{sym}} | \Psi_{\nu} \rangle &= \mathcal{C} \mathcal{C}_{\nu} \mathcal{C}_{\nu-1} \cdot \frac{\nu}{\mathcal{C}_{\nu}^2} \langle \psi_{\nu-1}^1 | \mathcal{V}_1 | \psi_{\nu}^1 \rangle \\ &= \sqrt{\nu} \sqrt{\frac{[N_1 N_5 - (\nu - 1)(l + 1)]! [N_1 N_5 - (l + 1)]!}{[N_1 N_5 - \nu(l + 1)]! (N_1 N_5)!}} \langle \psi_{\nu-1}^1 | \mathcal{V}_1 | \psi_{\nu}^1 \rangle \end{aligned} \quad (7.11)$$

## 7.5 The large $N_1 N_5$ limit

We are ultimately interested in the limit of large  $N_1 N_5$ . Then we have

$$\frac{[N_1 N_5 - (\nu - 1)(l + 1)]!}{[N_1 N_5 - \nu(l + 1)]!} \longrightarrow (N_1 N_5)^{l+1} \quad \frac{[N_1 N_5 - (l + 1)]!}{(N_1 N_5)!} \longrightarrow (N_1 N_5)^{-(l+1)}, \quad (7.12)$$

which gives

$$\langle \Psi_{\nu-1} | \mathcal{V}_{\text{sym}} | \Psi_{\nu} \rangle \longrightarrow \sqrt{\nu} \langle \psi_{\nu-1}^1 | \mathcal{V}_1 | \psi_{\nu}^1 \rangle. \quad (7.13)$$

The prefactor  $\sqrt{\nu}$  gives a ‘Bose enhancement’ effect which tells us that if we start with  $\nu$  quanta, the amplitude to emit another quantum is amplified by a factor  $\sqrt{\nu}$  (compared to the case when there was only one quantum). This gives an enhancement  $\nu$  in the probability, which just tells us that if we start with  $\nu$  quanta in the initial state, then the rate of emission is proportional to  $\nu$ .

## 8. The Rate of Emission

We now put together all the computations of the above sections to get the emission rate for a quantum from the excited CFT state. We need to do the following:

- (i) We use (7.13) to relate the decay amplitude for one  $(l + 1)$ -permutation to the amplitude with all the required symmetrizations put in

$$\langle \Psi_0 | \mathcal{V}_{\text{sym}} | \Psi_1 \rangle = \sqrt{\nu} \langle \psi_0^1 | \mathcal{V}_1 | \psi_1^1 \rangle. \quad (8.1)$$

- (ii) From Equation (6.8), we have the decay amplitude for a given  $l + 1$ -permutation (we put the right sector back in now):

$$\begin{aligned} \langle \psi_0^1 | \mathcal{V}_1 | \psi_1^1 \rangle &= \mathcal{A}^{\bar{ij}}(z, \bar{z}) \\ &= (-1)^{k+\bar{k}} \sqrt{\binom{N+l+1}{N} \binom{\bar{N}+l+1}{\bar{N}}} z^{-(\alpha+1)\frac{l}{2}+\alpha k-N-1} \bar{z}^{-(\bar{\alpha}+1)\frac{l}{2}+\bar{\alpha}\bar{k}-\bar{N}-1} \end{aligned} \quad (8.2)$$

We now rotating back to Lorentzian signature and replacing  $\tau, \sigma$  by the physical  $(t, y)$  coordinates. Note that we are still working with a CFT with spatial section of ‘unit size’ where the spatial circle has length  $(2\pi)$ . The ‘unit-sized’ amplitude is thus

$$\begin{aligned} \mathcal{A}_{\text{unit}}^{\bar{ij}}(t, y) &= (-1)^{k+\bar{k}} \sqrt{\binom{N+l+1}{N} \binom{\bar{N}+l+1}{\bar{N}}} \\ &\times e^{\frac{i}{R}(-(\alpha+1)\frac{l}{2}+\alpha k-N-1)(y+t)} e^{-\frac{i}{R}(-(\bar{\alpha}+1)\frac{l}{2}+\bar{\alpha}\bar{k}-\bar{N}-1)(y-t)}. \end{aligned} \quad (8.3)$$

- (iii) From Equation (8.3) or by comparing the initial and final states, we can read off

$$\begin{aligned} E_0 &= \frac{1}{R} [(\alpha + \bar{\alpha} + 2)\frac{l}{2} - \alpha k - \bar{\alpha}\bar{k} + N + \bar{N} + 2] = \frac{1}{R} [2l - k - \bar{k} + N + \bar{N} + 2] \\ \lambda_0 &= \frac{1}{R} [-(\alpha - \bar{\alpha})\frac{l}{2} + \alpha k - \bar{\alpha}\bar{k} - N + \bar{N}] = \frac{1}{R} [k - \bar{k} - N + \bar{N}], \end{aligned} \quad (8.4)$$

where we have set  $\alpha = \bar{\alpha} = 1$  for the physical process of interest. We also can determine the ‘unit-sized’ amplitude with the position dependence removed,

$$\langle f | \mathcal{V} | i \rangle_{\text{unit}} = \mathcal{A}_{\text{unit}}^{\bar{ij}}(0, 0) = (-1)^{k+1} \sqrt{\nu} \sqrt{\binom{N+l+1}{N} \binom{\bar{N}+l+1}{\bar{N}}}. \quad (8.5)$$

Putting this into Equation (2.65), one finds the final emission rate

$$\frac{d\Gamma}{dE} = \nu \frac{2\pi}{2^{2l+1} l!^2} \frac{(Q_1 Q_5)^{l+1}}{R^{2l+3}} (E^2 - \lambda^2)^{l+1} \binom{N+l+1}{N} \binom{\bar{N}+l+1}{\bar{N}} \delta_{\lambda, \lambda_0} \delta(E - E_0). \quad (8.6)$$

This is the emission rate for one of  $\nu$  excitations in the CFT to de-excite and emit a super-gravity particle with energy  $E_0$ ,  $S^1$ -momentum  $\lambda_0$ , and angular momentum

$$\begin{aligned} m_\psi &= -(m + \bar{m}) = -l + k + \bar{k} \\ m_\phi &= m - \bar{m} = \bar{k} - k. \end{aligned} \quad (8.7)$$

The angular momentum can be read off from the angular momentum of the NS sector initial state, or the difference in angular momentum between the initial and final physical states.

The expression for the emission rate obtained above matches the one obtained in [24] where it is given in a slightly different form. There the expression for a minimally coupled scalar to be absorbed into the geometry and re-emerge is given in Equation (5.34) along with



the time of travel in Equation (5.33). The total probability is the product of the probabilities to be absorbed and to re-emerge, which are equal. Therefore, the rate of emission is the square root of the total probability, with the energy and other quantum numbers taking the corresponding values for excitations in the background, divided by the time of travel. This expression is seen to match the emission rate obtained above.

## 9. Emission from nonextremal microstate

From a physics point of view the emission computed above corresponds to a very simple process. We take an extremal 2-charge D1D5 microstate, excite it by adding a quantum, and compute the rate at which the state de-excites by emitting this quantum.

But this same computation can be slightly modified to obtain the emission rate for a more interesting physical process. We start with a nonextremal D1D5 microstate which has a large energy above extremality. This particular microstate is obtained by taking an extremal D1D5 microstate and performing a spectral flow on both the left and right moving sectors. Such a spectral flow adds fermionic excitations to *every* component string. Thus we get a large energy above extremality, not just the energy of one nonextremal quantum as was the case with our earlier computations [23, 24, 12].

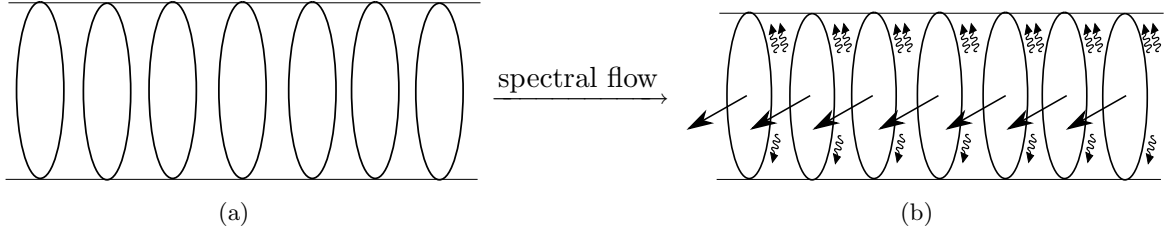
This nonextremal state emits radiation, and we wish to compute the rate of emission after  $\nu$  quanta have been emitted. We again get the ‘Bose enhancement’ like that in (7.13), so that the rate of emission keeps increasing as more quanta are emitted. In [14] it was shown that the resulting decay behavior is exactly the Hawking radiation expected from this particular microstate. But the computation of [14] was restricted to certain choices of spins and excitation level  $N = 0$  for the emitted quantum; now we are able to get a general expression for all values of spins and  $N$ .

### 9.1 The CFT process

As discussed earlier, the physical D1D5 system is in the Ramond sector. We can relate Ramond sector states to NS sector states by spectral flow. Recall that under spectral flow the dimensions and charges change as follows:

$$\begin{aligned} h' &= h + \alpha m + \frac{c\alpha^2}{24} \\ m' &= m + \frac{c\alpha}{12}. \end{aligned} \tag{9.1}$$

If we start with the NS vacuum  $|\emptyset\rangle_{NS}$  and spectral flow by  $\alpha = 1$ , we reach the Ramond vacuum state with  $h = \frac{c}{24}$ . But we can also reach a Ramond state by spectral flow by  $\alpha = 3, 5, \dots$ , which are excited states with energy more than the energy of the Ramond vacua. Let us take our initial state in the Ramond sector to be the state obtained by spectral flow of  $|\emptyset\rangle_{NS}$  by  $\alpha = 2n + 1$  on the left and  $\bar{\alpha} = 2\bar{n} + 1$  on the right. The spectral flow adds fermions to the left and right sectors, raising the level of the Fermi sea on both these sectors. Thus we get an excited state of the D1D5 system, which we depict in Fig. 4(b).



**Figure 4:** (a) The NS vacuum state in the CFT and (b) the CFT state after spectral flow. The arrows at the center of the circle indicate the ‘base spin’ of component strings in the Ramond sector. The wavy arrows on top (bottom) of the strands represent fermionic excitations in the left (right) sector.

The vertex operator we have constructed can twist together  $l + 1$  copies of the CFT. In our earlier computation, we started with a set of twisted copies, and the vertex operator ‘untwisted’ these, leading to a final state with no twists. This time the initial state has all copies of the CFT ‘untwisted’, but these copies are all in an excited state. The vertex operator can therefore twist together  $l + 1$  copies, leading to a twisted component string in the final state. Even though twisting a set of strings increases the energy, this component string in the final state can have lower energy than the strings in the initial state because of the fermionic excitations present on the initial component strings. The energy difference between the initial and final states is the energy of the emitted supergravity particle.

Let us now set up the CFT computation needed for this process. We observe that the amplitude can be obtained in a simple way from the amplitude that we have already computed.

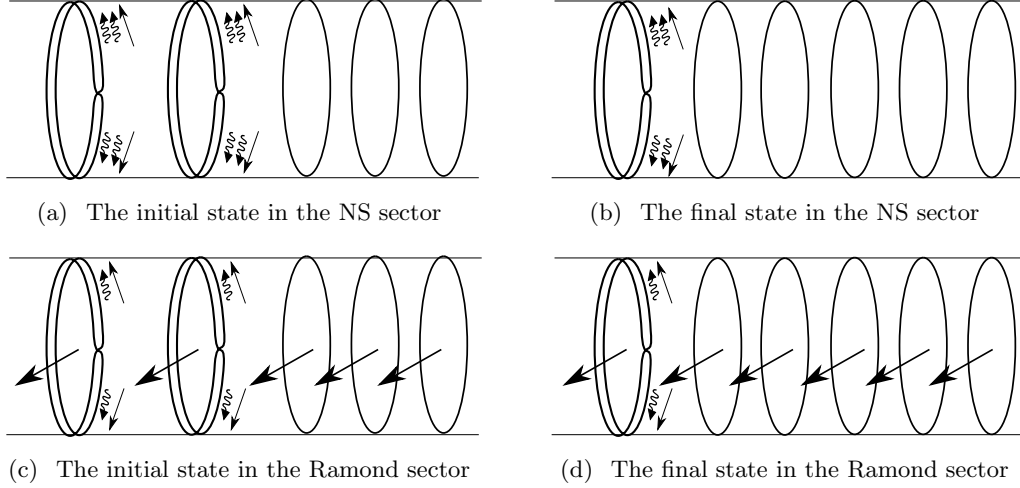
## 9.2 The initial state

As before, we do all our computations in the NS sector. If we spectral flow the starting state depicted in Fig. 4(b) by  $-(2n + 1)$  units, we arrive at the NS vacuum  $|\emptyset\rangle_{NS}$  depicted in Fig. 4(a). It may appear that this vacuum state cannot lead to any emission, but recall that we have used spectral flow only as a technical trick; the actual initial state has a much higher energy, and indeed leads to emission.

If we wanted to start with this state and proceed with the computation we would set  $|i'\rangle = |\emptyset\rangle_{NS}$ . But we instead look at a slightly more general situation where  $\nu - 1$  quanta have already been emitted. In this case, the initial state looks like the one depicted in Fig. 6(c), where  $\nu - 1$  sets of  $l + 1$  copies have already been twisted together.

This may look like a complicated initial state, but we look only at a specific amplitude: the amplitude for emission of a further quantum of the same kind as the quanta already present. This process therefore requires us to take  $l + 1$  of the *untwisted* copies of the CFT, and use the vertex operator to twist them together. The other copies of the CFT are unaffected by the vertex operator. Thus, for the purposes of computing the amplitude, the initial state of the  $l + 1$  copies of interest is

$$|i'\rangle = |\emptyset\rangle_{NS}. \quad (9.2)$$



**Figure 5:** The initial and final states for the emission process discussed in Sections 4, 6, and 8. The pictures correspond to  $\nu = 2$  and  $l = 1$  emission. The straight arrows pointing up (down) on the loops indicate bosonic excitations in the left (right) sector.

### 9.3 The final state

The final state is determined by the fact that we are looking for the amplitude to transition to a supergravity state, and we have a unique supergravity excitation with given twist and angular quantum numbers. Working again in the NS sector, arrived at by spectral flow by  $-(2n + 1)$  units, we get

$$|f'\rangle = |\phi_{N+1}^{\frac{l}{2}, \frac{l}{2}-k}\rangle; \quad (9.3)$$

the initial state of our previous calculation.

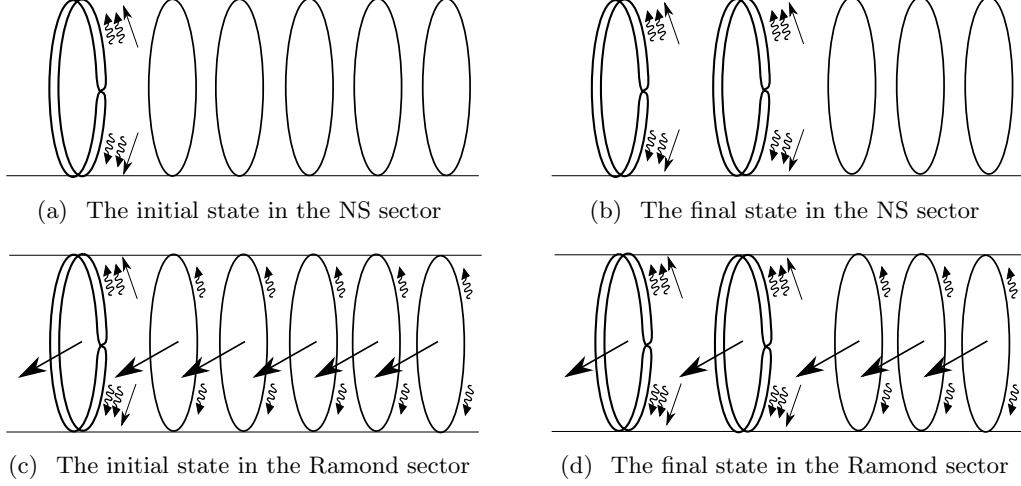
### 9.4 The vertex operator

The vertex operator is independent of the states it acts on. It is completely determined by the supergravity scalar to which it couples in Equation (2.52).

We now see that the present process is similar to the amplitude we computed earlier if we reverse the direction of time  $\tau$ . That is, the initial state now is untwisted, while in our earlier computation the *final* state was untwisted. The vertex operator then leads to a final twisted state, and since there is a unique supergravity state with given quantum numbers, we can write down this state.

### 9.5 Relating the emission amplitude to the earlier computed amplitude

We term the supergravity excitation emission in the previous sections ‘untwisting’ emission since a twisted component string in the initial state ‘untwists’ under the action of the vertex operator and leads to a final state with no twists. We call the emission of the present section



**Figure 6:** The initial and final states for the nonextremal emission process discussed in Section 9. The specific case depicted is  $\nu = 2$  and  $l = 1$ .

‘twisting’ emission since the initial state has no twists, and the vertex operator leads to a twisted component string in the final state.

By comparing the initial and final states of the two process, we immediately see that the current NS sector twisting amplitude is simply the Hermitian conjugate of the previous NS sector untwisting amplitude,

$$\mathcal{A}'_{l,m_\psi,m_\phi}{}^{\text{twisting}}(t,y) = [\mathcal{A}'_{l,-m_\psi,-m_\phi}{}^{\text{untwisting}}(-t,y)]^\dagger, \quad (9.4)$$

with flipped  $SO(4)_E$  charges and reversed time. In the complex plane, this statement becomes

$$\mathcal{A}'_{l,m_\psi,m_\phi}{}^{\text{twisting}}(z,\bar{z}) = [\mathcal{A}'_{l,-m_\psi,-m_\phi}{}^{\text{untwisting}}(\frac{1}{\bar{z}},\frac{1}{z})]^\dagger. \quad (9.5)$$

To see the above relation explicitly, consider the Hermitian conjugate of the previous, untwisting amplitude:

$$\begin{aligned} [\mathcal{A}'_{l,m_\psi,m_\phi}{}^{\text{untwisting}}(z,\bar{z})]^\dagger &= [|z|^{l+2} \langle f' | \mathcal{V}_{l,m_\psi,m_\phi}(z,\bar{z}) | i' \rangle]^\dagger \\ &= |z|^{l+2} \langle i' | [\mathcal{V}_{l,m_\psi,m_\phi}(z,\bar{z})]^\dagger | f' \rangle \\ &= \frac{1}{|z|^{l+2}} \langle i' | \mathcal{V}_{l,-m_\psi,-m_\phi}(\frac{1}{\bar{z}},\frac{1}{z}) | f' \rangle, \end{aligned} \quad (9.6)$$

where the  $i'$  and  $f'$  are from the previous calculation. The amplitude we now wish to compute is (in terms of the previous calculation’s states)

$$\mathcal{A}'_{l,m_\psi,m_\phi}{}^{\text{twisting}}(z,\bar{z}) = |z|^{l+2} \langle i' | \mathcal{V}_{l,m_\psi,m_\phi}(z,\bar{z}) | f' \rangle \quad (9.7)$$

Comparing these two expressions, one arrives at Equation (9.5).

From Equation (6.8), we have

$$\mathcal{A}'^{\text{untwisting}}(z, \bar{z}) = (-1)^{k+\bar{k}} \sqrt{\binom{N+l+1}{N} \binom{\bar{N}+l+1}{\bar{N}}} z^{-\frac{l}{2}-N-1} \bar{z}^{-\frac{l}{2}-\bar{N}-1}, \quad (9.8)$$

and so using Equation (9.5) gives

$$\mathcal{A}'^{\text{twisting}} = (-1)^{k+\bar{k}} \sqrt{\binom{N+l+1}{N} \binom{\bar{N}+l+1}{\bar{N}}} z^{\frac{l}{2}+N+1} \bar{z}^{\frac{l}{2}+\bar{N}+1}. \quad (9.9)$$

Note that this amplitude is in the NS sector, and before we can use it to get emission we have to spectral flow it to the Ramond sector. Spectral flowing by  $\alpha = 2n + 1$  units to the Ramond sector gives

$$\begin{aligned} \mathcal{A}_{\text{twisting}}^{(\alpha, \nu)} &= z^{-\alpha(\frac{l}{2}-k)} \bar{z}^{-\bar{\alpha}(\frac{l}{2}-\bar{k})} \mathcal{A}'^{\nu}_{\text{twisting}} \\ &= (-1)^{k+\bar{k}} \sqrt{\nu} \sqrt{\binom{N+l+1}{N} \binom{\bar{N}+l+1}{\bar{N}}} z^{\frac{l}{2}+N+1-\alpha(\frac{l}{2}-k)} \bar{z}^{\frac{l}{2}+\bar{N}+1-\bar{\alpha}(\frac{l}{2}-\bar{k})} \end{aligned} \quad (9.10)$$

where we have put the combinatoric factor in as well. Note that the combinatorics work out the same as before since the combinatorics cannot be affected by Hermitian conjugation; however, the interpretation is different. The initial state starts with  $\nu - 1$  sets of  $(l+1)$ -twisted component strings, while the final state has  $\nu$   $(l+1)$ -twisted component strings. Therefore, if at some initial time all of the strands were untwisted and each twist corresponds to an emitted supergravity particle, then the above is the amplitude for the emission of the  $\nu$ th particle.

Comparing Equation (9.10) with Equation (8.2), we see that the amplitudes agree except for the power of  $z$ , which is different because of the different energies of the concerned states in the two processes. Thus we can immediately write down the emission rate for the  $\nu$ th particle from the cap into the flat space

$$\frac{d\Gamma}{dE} = \nu \frac{2\pi}{2^{2l+1} l!^2} \frac{(Q_1 Q_5)^{l+1}}{R^{2l+3}} (E^2 - \lambda^2)^{l+1} \binom{N+l+1}{N} \binom{\bar{N}+l+1}{\bar{N}} \delta_{\lambda, \lambda_0} \delta(E - E_0) \quad (9.11)$$

where

$$\begin{aligned} E_0 &= \frac{1}{R} [(\alpha + \bar{\alpha} - 2)\frac{l}{2} - \alpha k - \bar{\alpha} \bar{k} - N - \bar{N} - 2] \\ \lambda_0 &= \frac{1}{R} [-(\alpha - \bar{\alpha})\frac{l}{2} + \alpha k - \bar{\alpha} \bar{k} + N - \bar{N}]. \end{aligned} \quad (9.12)$$

For sufficiently large  $\alpha$  or  $\bar{\alpha}$ ,  $E_0$  is positive and the physical process is emission and not absorption. In taking the hermitian conjugate we have flipped the angular momentum of the emitted particle from that in Equation (8.7); therefore, the above emission rate is for

$$m_\psi = l - k - \bar{k} \quad m_\phi = k - \bar{k}. \quad (9.13)$$

One can check that the emission rate above agrees with the emission rate from the gravity dual [13, 14]. Such a check was carried out in [14] only for states with excitation level  $N = 0$ , because it was not clear how to construct the initial state for  $N > 0$  in the effective string description of the D1D5 bound state. With our present construction of states and vertex operators in the orbifold CFT, we can compute amplitudes for emission of supergravity quanta from all initial states containing supergravity excitations.

## 10. Discussion

The D1D5 system is described by a 1+1 dimensional CFT, just as D3 branes give rise to a 3+1 dimensional Yang-Mills CFT. The Yang-Mills case has a simple ‘free’ point where the coupling vanishes and computations are simple. The analog of the ‘free’ point for the 1+1 CFT is expected to be the orbifold CFT.

Even though the orbifold CFT is expected to be ‘simple’, one faces the complication that the orbifold group is nonabelian (it is the permutation group  $S_{N_1 N_5}$ ). Thus computations of 3-point functions take more effort than would be required in a free theory or an orbifold theory with an abelian orbifold group [57, 64, 65, 66].

Our goal is to relate CFT computations to gravity computations, particularly those related to black holes. In this paper we have made definitive progress toward this end. First we set up a general formalism relating amplitudes computed in the CFT to scattering amplitudes for quanta incident from flat infinity. This is important because the quanta coming from flat infinity do not directly reach the  $AdS$  boundary; they have to pass through an ‘intermediate region’, which introduces a deformation that we must consider in general. However, for minimal scalars, which we considered in detail, it turns out that the intermediate region does not introduce a deformation.

Next, we constructed the vertex operators that describe the emission and absorption of supergravity particles. While some supergravity quanta are described by chiral primary operators, others arise from the descendants of these operators obtained by acting with the anomaly-free part of the chiral algebra. The minimal scalars that we considered arise as descendants, and we constructed these operators and normalized them.

We then computed the amplitude of emission from a simple excited state of the D1D5 system. This computation is performed most easily by spectral flowing the system to the NS sector. We examined how spectral flow changed the states and vertex operator, we performed the computation in the NS sector, and then we spectral flowed back to the Ramond sector where the physical theory actually lives.

To finish this computation we had to take into account combinatoric factors that count the different ways in which we can permute the copies of the orbifold theory involved in the interaction. Through these steps, we obtained the rate of emission of quanta from the excited D1D5 CFT state. We then compared the rate to the known result in the dual gravity description, and found exact agreement. It is true that the CFT and gravity theories

hold at different couplings, but it is well known that simple low energy processes involving supergravity quanta often agree at leading order between the two descriptions.

Finally we noted that the CFT amplitude we computed above could be related to the CFT amplitude for a different process: emission from a highly excited state obtained by spectral flow from the CFT vacuum state. Again the emission rate computed from the CFT was found to agree exactly with the emission rate obtained in the gravity description [13].

Note that once we normalize the CFT 2-point function to agree with the gravity 2-point function, we will get an agreement between the CFT and gravity for all processes that involve only the 2-point function. We have made such a normalization using the short distance behavior of the 2-point function in the CFT, and then used this to compute emission: a process that depends on ‘long distance physics’ because it depends on the length of the effective string. Our physics goal is to compare the emission rate found from ergoregion instability in [13] with a CFT computation; which is what we achieve in the end.

Such comparisons of emission rates between CFT and gravity computations have been performed many times in the past [4, 5, 6, 14], but the vertex leading to emission was previously modeled somewhat heuristically. In particular, in [14] the heuristic vertex was used to get the emission for supergravity quanta with  $N = 0$  (the lowest energy state in a given angular momentum sector), but it was not clear how to make the CFT state for higher  $N$ . In the present paper we have constructed the CFT state for all  $N$ , so emissions of all supergravity quanta can be computed. It is hoped that these explicit constructions will allow us to perform a large set of further computations, including those that lead to deformations away from the orbifold point.

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## A. Notation and conventions for the orbifolded CFT

Here, we carefully define the notation and conventions used throughout this paper, and to be used in future work. The system we describe lives in  $M_{4,1} \times T^4 \times S^1$ . We restrict our attention to the case where the compact space is a torus, although one may also consider K3.

The base space of the CFT is the  $S^1$  and time, with fields living in the orbifolded target space:  $(T^4)^{N_1 N_5} / S_{N_1 N_5}$ . The expressions in this appendix are exclusively given for the complex plane.

## A.1 Symmetries and indices

The symmetries of our theory are  $SU(2)_L \times SU(2)_R$  and the  $SO(4)_I \simeq SU(2)_1 \times SU(2)_2$  rotations of the torus. Indices correspond to the following representations

$$\begin{array}{llll} \alpha, \beta & \text{doublet of } SU(2)_L & \dot{\alpha}, \dot{\beta} & \text{doublet of } SU(2)_R \\ A, B & \text{doublet of } SU(2)_1 & \dot{A}, \dot{B} & \text{doublet of } SU(2)_2 \\ i, j & \text{vector of } SO(4). & & \end{array}$$

One can project vectors of  $SO(4)$  into the doublets of  $SU(2)_1 \times SU(2)_2$ , using the usual Pauli spin matrices and the identity matrix

$$(\sigma^i)^{\dot{A}A}, \quad \sigma^4 = i\mathbb{1}_2.$$

The indices are such that, for instance,  $(\sigma^2)^{\dot{2}1} = i$ .

We use indices  $a, b, c = 1, 2, 3$  for the triplet of any  $SU(2)$ . Their occurrence is rare enough that which  $SU(2)$  is being referred to should be unambiguous. Note that the  $SU(2)$  generators  $(\sigma^a)_\alpha{}^\beta$  naturally come with one index raised and one index lowered. On the other hand the Clebsch-Gordan coefficients to project a vector of  $SO(4)$  into two  $SU(2)$ 's naturally come with both indices raised (or lowered), as above.

We raise and lower all  $SU(2)$  doublet indices in the same way so that

$$\epsilon_{\alpha\beta} v^\beta = v_\alpha \quad v^\alpha = \epsilon^{\alpha\beta} v_\beta, \quad (\text{A.1})$$

where

$$\epsilon_{12} = -\epsilon_{21} = \epsilon^{21} = -\epsilon^{12} = 1, \quad (\text{A.2})$$

and therefore

$$\epsilon_{\alpha\beta} \epsilon^{\beta\gamma} = \delta_\alpha^\gamma. \quad (\text{A.3})$$

## A.2 Field content

The bosonic field content of each copy of the CFT consists of a vector of  $SO(4)_I$ ,  $X^i(z, \bar{z})$ , giving the position of that effective string in the torus. The fermions on the left sector have indices in  $SU(2)_L \times SU(2)_2$ , while the fermions in the right sector have indices in  $SU(2)_R \times SU(2)_2$ :

$$\psi^{\alpha\dot{A}}(z) \quad \bar{\psi}^{\dot{\alpha}\dot{A}}(\bar{z}). \quad (\text{A.4})$$

These fermions are complex, so there are two complex fermions and their Hermitian conjugates in the left sector and two complex fermions and their Hermitian conjugates in the right sector.

Note that we use the abbreviated notation

$$[X]^{\dot{A}A} = X^i(\sigma^i)^{\dot{A}A}. \quad (\text{A.5})$$



### A.3 Currents

The holomorphic currents of our theory that form a closed OPE algebra are an  $SU(2)_L$  current,  $J^a(z)$ ; the supersymmetry currents,  $G^{\alpha A}(z)$ ; and the stress-energy  $T(z)$ . The right sector has the corresponding anti-holomorphic currents. Obviously, in this case, the index  $a$  on  $J$  transforms in  $SU(2)_L$ .

For each copy of the CFT, the currents are realized in terms of the fields as

$$J^a(z) = \frac{1}{4} \epsilon_{\dot{A}\dot{B}} \psi^{\alpha\dot{A}} \epsilon_{\alpha\beta} (\sigma^{*a})^\beta_\gamma \psi^{\gamma\dot{B}} \quad (\text{A.6a})$$

$$G^{\alpha A}(z) = \psi^{\alpha\dot{A}} [\partial X]^{\dot{B}A} \epsilon_{\dot{A}\dot{B}} \quad (\text{A.6b})$$

$$T(z) = \frac{1}{4} \epsilon_{\dot{A}\dot{B}} \epsilon_{AB} [\partial X]^{\dot{A}A} [\partial X]^{\dot{B}B} + \frac{1}{2} \epsilon_{\alpha\beta} \epsilon_{\dot{A}\dot{B}} \psi^{\alpha\dot{A}} \partial \psi^{\beta\dot{B}}. \quad (\text{A.6c})$$

Note that the  $SO(4)_I$  of the torus is an outer automorphism and so while we can make a generator that acts appropriately on the fermions we cannot make one that also acts appropriately on the bosons.

### A.4 Hermitian conjugation

Because we work in a Euclidean time formalism, one must address Hermitian conjugation carefully so that it is consistent with the physical, real-time formalism.

A quasi-primary field of weight  $(\Delta, \bar{\Delta})$  is Hermitian conjugated as [67]

$$[\mathcal{O}(z, \bar{z})]^\dagger = \bar{z}^{-2\Delta} z^{-2\bar{\Delta}} \mathcal{O}^\dagger\left(\frac{1}{\bar{z}}, \frac{1}{z}\right), \quad (\text{A.7})$$

where  $\mathcal{O}^\dagger(z, \bar{z})$  has the opposite charges under  $SU(2)_L \times SU(2)_R \times SO(4)_I$ .

The fermions Hermitian conjugate as

$$(\psi^{\alpha\dot{A}})^\dagger(z) = -\epsilon_{\alpha\beta} \epsilon_{\dot{A}\dot{B}} \psi^{\beta\dot{B}}(z) = -\psi_{\alpha\dot{A}}(z). \quad (\text{A.8})$$

This reality condition ensures that there are only four real degrees of freedom in both the left and right sectors. The specific sign can be determined from the basic fermion correlator and demanding a positive-definite norm.

The bosons conjugate as

$$(X^i)^\dagger(z, \bar{z}) = X^i(z, \bar{z}) \quad ([X]^{\dot{A}A})^\dagger(z, \bar{z}) = -\epsilon_{\dot{A}\dot{B}} \epsilon_{AB} [X]^{\dot{B}B}(z, \bar{z}). \quad (\text{A.9})$$

The stress energy tensor and the  $SU(2)_L$  current are both Hermitian and so conjugate trivially; whereas, the supercurrents conjugate as

$$(G^{\alpha A})^\dagger(z) = -\epsilon_{\alpha\beta} \epsilon_{AB} G^{\beta B}(z). \quad (\text{A.10})$$

Again, the specific sign is determined by requiring the norm to be positive-definite.

The Ramond vacua conjugate as

$$(|\varnothing\rangle_R^\alpha)^\dagger = {}^R_\alpha \langle \varnothing| \quad {}^R_\alpha \langle \varnothing| \varnothing \rangle_R^\beta = \delta_\alpha^\beta. \quad (\text{A.11})$$

## A.5 OPE

We normalize the fields so that the basic correlators are

$$\langle X^i(z)X^j(w) \rangle = -2\delta^{ij} \log |z - w| \quad (\text{A.12a})$$

$$\langle \psi^{\alpha\dot{A}}(z)\psi^{\beta\dot{B}}(w) \rangle = -\frac{\epsilon^{\alpha\beta}\epsilon^{\dot{A}\dot{B}}}{z - w}, \quad (\text{A.12b})$$

where it is also useful to note that Equation (A.12a) implies

$$\langle [X]^{\dot{A}\dot{A}}(z)[X]^{\dot{B}\dot{B}}(w) \rangle = 4\epsilon^{\dot{A}\dot{B}}\epsilon^{AB} \log |z - w|. \quad (\text{A.13})$$

From which, the commonly used

$$[\partial X(z)]^{\dot{A}\dot{A}}[\partial X(w)]^{\dot{B}\dot{B}} \sim 2\frac{\epsilon^{\dot{A}\dot{B}}\epsilon^{AB}}{(z - w)^2}, \quad (\text{A.14})$$

immediately follows.

The OPE current algebra for a single copy of the  $\mathcal{N} = 4$  CFT is

$$J^a(z)J^b(w) \sim \frac{c}{12} \frac{\delta^{ab}}{(z - w)^2} + i\epsilon^{ab}{}_c \frac{J^c(w)}{z - w} \quad (\text{A.15a})$$

$$J^a(z)G^{\alpha A}(w) \sim \frac{1}{2}(\sigma^{*a})^\alpha{}_\beta \frac{G^{\beta A}(w)}{z - w} \quad (\text{A.15b})$$

$$G^{\alpha A}(z)G^{\beta B}(w) \sim -\frac{2c}{3} \frac{\epsilon^{AB}\epsilon^{\alpha\beta}}{(z - w)^3} + 2\epsilon^{AB}\epsilon^{\beta\gamma}(\sigma^{*a})^\alpha{}_\gamma \left[ \frac{2J^a(w)}{(z - w)^2} + \frac{\partial J^a(w)}{z - w} \right] - 2\epsilon^{AB}\epsilon^{\alpha\beta} \frac{T(w)}{z - w} \quad (\text{A.15c})$$

$$T(z)J^a(w) \sim \frac{J^a(w)}{(z - w)^2} + \frac{\partial J^a(w)}{z - w} \quad (\text{A.15d})$$

$$T(z)G^{\alpha A}(w) \sim \frac{\frac{3}{2}G^{\alpha A}(w)}{(z - w)^2} + \frac{\partial G^{\alpha A}(w)}{z - w} \quad (\text{A.15e})$$

$$T(z)T(w) \sim \frac{c}{2} \frac{1}{(z - w)^4} + 2\frac{T(w)}{(z - w)^2} + \frac{\partial T(w)}{z - w}, \quad (\text{A.15f})$$

which agrees with the above correlators for  $c = 6$ .

For convenient reference, we include the OPEs of the currents with the basic primary fields,  $\partial X$  and  $\psi$ :

$$J^a(z)\psi^{\alpha\dot{A}}(w) \sim \frac{1}{2}(\sigma^{*a})^\alpha{}_\beta \frac{\psi^{\beta\dot{A}}(w)}{z - w} \quad (\text{A.16a})$$

$$G^{\alpha A}(z)[\partial X(w)]^{\dot{B}\dot{B}} \sim 2\epsilon^{AB} \left( \frac{\psi^{\alpha\dot{B}}(w)}{(z - w)^2} + \frac{\partial \psi^{\alpha\dot{B}}(w)}{z - w} \right) \quad (\text{A.16b})$$

$$G^{\alpha A}(z)\psi^{\beta\dot{A}}(w) \sim \epsilon^{\alpha\beta} \frac{[\partial X(w)]^{\dot{A}\dot{A}}}{z - w} \quad (\text{A.16c})$$

$$T(z)[\partial X(w)]^{\dot{A}A} \sim \frac{[\partial X(w)]^{\dot{A}A}}{(z-w)^2} + \frac{[\partial^2 X(w)]^{\dot{A}A}}{z-w} \quad (\text{A.16d})$$

$$T(z)\psi^{\alpha\dot{A}}(w) \sim \frac{\frac{1}{2}\psi^{\alpha\dot{A}}(w)}{(z-w)^2} + \frac{\partial\psi^{\alpha\dot{A}}(w)}{z-w}. \quad (\text{A.16e})$$

## A.6 Mode algebra

We define the modes corresponding to the above currents according to their weight,  $\Delta$ , by

$$\begin{aligned} \mathcal{O}_m &= \oint \frac{dz}{2\pi i} \mathcal{O}(z) z^{\Delta+m-1} \\ \mathcal{O}(z) &= \sum_m \mathcal{O}_m z^{-(\Delta+m)}. \end{aligned} \quad (\text{A.17})$$

The weight may be read off from the OPE of the current with the stress-energy tensor. Fermionic currents have half-integer weight. In the NS sector, fermions are periodic in the plane and therefore we need integer powers of  $z$ . This means the fermionic currents have modes labeled by half-integer  $m$ .

Using the OPE current algebra above, one finds that the modes form an algebra:

$$[J_m^a, J_n^b] = \frac{c}{12} m \delta^{ab} \delta_{m+n,0} + i \epsilon^{ab}{}_c J_{m+n}^c \quad (\text{A.18a})$$

$$[J_m^a, G_n^{\alpha A}] = \frac{1}{2} (\sigma^{*a})^\alpha{}_\beta G_{m+n}^{\beta A} \quad (\text{A.18b})$$

$$\{G_m^{\alpha A}, G_n^{\beta B}\} = -\frac{c}{3} (m^2 - \frac{1}{4}) \epsilon^{AB} \epsilon^{\alpha\beta} \delta_{m+n,0} + 2(m-n) \epsilon^{AB} \epsilon^{\beta\gamma} (\sigma^{*a})^\alpha{}_\gamma J_{m+n}^a - 2\epsilon^{AB} \epsilon^{\alpha\beta} L_{m+n} \quad (\text{A.18c})$$

$$[L_m, J_n^a] = -n J_{m+n}^a \quad (\text{A.18d})$$

$$[L_m, G_n^{\alpha A}] = (\frac{m}{2} - n) G_{m+n}^{\alpha A} \quad (\text{A.18e})$$

$$[L_m, L_n] = c \frac{m^3 - m}{12} \delta_{m+n,0} + (m-n) L_{m+n}. \quad (\text{A.18f})$$

The infinite-dimensional algebra has a finite, anomaly-free subalgebra which is of primal importance for the AdS-CFT correspondence. The anomaly-free subalgebra has a basis of  $\{J_0^a, G_{\pm\frac{1}{2}}^{\alpha A}, L_0, L_{\pm 1}\}$ . The smaller subalgebra spanned by  $\{J_0^3, L_0\}$  is the Cartan subalgebra, which means we may label states and operators by their charge  $m$  and their weight  $h$ .

For reference, we provide the mode algebra of the two canonical primary fields. The  $\partial X$ 's modes are  $\alpha_n$ .

$$[\alpha_m^{\dot{A}A}, \alpha_n^{\dot{B}B}] = 2m \epsilon^{\dot{A}\dot{B}} \epsilon^{AB} \delta_{n+m,0} \quad (\text{A.19a})$$

$$\{\psi_m^{\alpha\dot{A}}, \psi_n^{\beta\dot{B}}\} = -\epsilon^{\alpha\beta} \epsilon^{\dot{A}\dot{B}} \delta_{m+n,0} \quad (\text{A.19b})$$

$$[J_m^a, \psi_n^{\alpha\dot{A}}] = \frac{1}{2} (\sigma^{*a})^\alpha{}_\beta \psi_{m+n}^{\beta\dot{A}} \quad (\text{A.19c})$$

$$[G_m^{\alpha A}, \alpha_n^{\dot{B}B}] = -2n \epsilon^{AB} \psi_{m+n}^{\alpha\dot{B}} \quad (\text{A.19d})$$

$$\{G_m^{\alpha A}, \psi_n^{\beta\dot{A}}\} = \epsilon^{\alpha\beta} \alpha_{m+n}^{\dot{A}A} \quad (\text{A.19e})$$

$$[L_m, \alpha_n^{\dot{A}A}] = -n\alpha_{m+n}^{\dot{A}A} \quad (\text{A.19f})$$

$$[L_m, \psi_n^{\alpha\dot{A}}] = -(\frac{m}{2} + n)\psi_{m+n}^{\alpha\dot{A}}. \quad (\text{A.19g})$$

### A.7 Useful identities

These identities are useful for relating vectors of  $SO(4)_I$  to tensors in  $SU(2)_1 \times SU(2)_2$ :

$$\epsilon_{\dot{A}\dot{B}}\epsilon_{AB}(\sigma^i)^{\dot{A}A}(\sigma^j)^{\dot{B}B} = -2\delta^{ij} \quad (\text{A.20a})$$

$$(\sigma^i)^{\dot{A}A}(\sigma^i)^{\dot{B}B} = -2\epsilon^{\dot{A}\dot{B}}\epsilon^{AB}. \quad (\text{A.20b})$$

It is useful to know how to relate the  $(+, -, 3)$  basis for the triplet of  $SU(2)$  to the  $(1, 2, 3)$  basis:

$$\delta^{++} = \delta^{--} = \delta_{++} = \delta_{--} = 0 \quad (\text{A.21a})$$

$$\delta^{+-} = \delta^{-+} = 2 \quad \delta_{+-} = \delta_{-+} = \frac{1}{2} \quad (\text{A.21b})$$

$$\epsilon^{+-3} = -2i \quad (\text{A.21c})$$

$$\sigma^+ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \quad \sigma^- = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}. \quad (\text{A.21d})$$

One can raise and lower the ‘3’ index with impunity.

### A.8 $n$ -twisted sector mode algebra

In the  $n$ -twisted sector, by which we mean modes whose contour orbits a twist operator, we can only define the modes by summing over all  $n$ -copies of the field. This allows us to define fractional modes. The modes are defined by [57]

$$\mathcal{O}_{\frac{m}{n}} = \oint_0 \frac{dz}{2\pi i} \sum_{k=1}^n \mathcal{O}^{(k)}(z) e^{2\pi i \frac{m}{n}(k-1)} z^{\Delta + \frac{m}{n} - 1}. \quad (\text{A.22})$$

One can confirm that the integrand is  $2\pi$ -periodic and therefore well-defined. If one lifts to a covering space using a map that locally behaves as

$$z = bt^n, \quad (\text{A.23})$$

then the mode in the base  $z$ -space can be related to a mode in the  $t$ -space:

$$\mathcal{O}_{\frac{m}{n}}^{(z)} = b^{\frac{m}{n}} n^{1-\Delta} \mathcal{O}_m^{(t)}, \quad (\text{A.24})$$

where  $\Delta$  is the weight of the field.

To compute a correlator in the twisted sector, one may either work in the base space with the summed-over-copies modes or one may work in the covering space with the opened-up mode. If one works in the base space, then one should use the algebra with the *total* central charge

$$c_{\text{tot.}} = nc; \quad (\text{A.25})$$

the algebra is otherwise unchanged. If one works in the covering space then one uses the central charge of a single copy of the CFT, but must remember to write all of the factors that come in lifting to the cover. These two methods give identical answers.

## A.9 Spectral flow

The  $\mathcal{N} = 4$  algebra is a vector space at every point  $z$  in the complex plane, spanned by the local operators  $\{J^a(z), G^{\alpha A}(z), T(z)\}$ . This vector space closes under the OPE. It is possible to make a  $z$ -dependent change of basis and preserve the algebra.

Making an  $SU(2)_L$  transformation in the ‘3’ direction to the local operators by an angle,  $\eta(z) = i\alpha \log z$ , at every point  $z$  is called ‘spectral flow’ by  $\alpha$  units. While this may look like a nontrivial transformation, the new algebra is isomorphic to the old algebra [61].

It is important to remember that  $\log z$  has a branch cut, which we put on the real axis for the following discussion. Let us suppose we start in the NS sector, where the local operators are periodic in the complex plane. Let us spectral flow the local operators by  $\alpha$  units. Suppose we start on the (positive imaginary side of the) positive real axis, where  $\eta = 0$  and the new operators are the same as the old operators. As we make a counter-clockwise circle in the complex plane, the angle between the old operators and the new operators increases. Across the branch cut on the real axis, where before the operators were continuous, now there is a large, finite  $SU(2)_L$  transformation.

More illustratively, consider how the fermions behave under spectral flow, as described above:

$$\psi^{\pm\dot{A}}(z) \mapsto \psi^{\pm\dot{A}'}(z) = e^{\pm\frac{i}{2}\eta(z)} \psi^{\pm\dot{A}}(z) = z^{\mp\frac{\alpha}{2}} \psi^{\pm\dot{A}}(z). \quad (\text{A.26})$$

We see that except for even  $\alpha$ , there is a branch cut. Moreover, if we spectral flow by an odd number of units, then the new operators  $\psi'(z)$  have the opposite periodicity from  $\psi(z)$ . In general, one expects that an operator with charge  $m$  under  $SU(2)_L$  transforms as

$$\mathcal{O}(z) \mapsto z^{-\alpha m} \mathcal{O}(z); \quad (\text{A.27})$$

however, the superconformal algebra and its  $SU(2)_L$  subalgebra, in particular, is anomalous which leads to nontrivial transformations of some operators.

Since the spectral flowed algebra and the original algebra are isomorphic, there is a bijective mapping from states living in the representations of one algebra to states living in the representation of its spectral flow. Since the NS sector and the R sector are related by spectral flow, we can map problems in one sector into problems in the other.

The operator which maps states into their spectral flow images, we call  $\mathcal{U}_\alpha$ ,

$$|\psi'\rangle = \mathcal{U}_\alpha |\psi\rangle. \quad (\text{A.28})$$

Formally, then, we may write the action of spectral flow on operators as

$$\mathcal{O}'(z) = \mathcal{U}_\alpha \mathcal{O}(z) \mathcal{U}_\alpha^{-1}, \quad (\text{A.29})$$

so that amplitudes are invariant under spectral flow. The spectral flow operator,  $\mathcal{U}_\alpha$  may be roughly defined as an ‘improper gauge transformation’ [61, 62, 63].

The spectral flow operator is most naturally defined in the context of bosonized fermions. We can bosonize the fermions as (conventions chosen to be consistent with [57])

$$\psi^{+1} = e^{-i\phi_6} \quad \psi^{+2} = e^{i\phi_5} \quad \psi^{-1} = e^{-i\phi_5} \quad \psi^{-2} = -e^{i\phi_6}, \quad (\text{A.30})$$

which gives the  $SU(2)_L$  current in the form<sup>6</sup>

$$J^3(z) = \frac{i}{2}(\partial\phi_5(z) - \partial\phi_6(z)) \quad J^+(z) = e^{-i\phi_6} e^{i\phi_5}(z) \quad J^-(z) = e^{-i\phi_5} e^{i\phi_6}(z). \quad (\text{A.31})$$

The fields  $\phi_5$  and  $\phi_6$  are the (holomorphic half of) real bosons normalized such that

$$\langle \phi_i(z) \phi_j(w) \rangle = -\delta_{ij} \log(z - w). \quad (\text{A.32})$$

They may be expanded as

$$\phi_i = q_i - \frac{i}{2} p_i \log z + (\text{modes}), \quad (\text{A.33})$$

where  $q_i$  and  $p_i$  are the zero-mode position and momentum which satisfy

$$[q_i, p_j] = i\delta_{ij}. \quad (\text{A.34})$$

With this bosonization, the spectral flow operator can be written as [63]

$$\mathcal{U}_\alpha = e^{i\alpha(q_5 - q_6)}. \quad (\text{A.35})$$

We see that spectral flow corresponds to increasing and decreasing the zero mode momentum of the fields  $\phi_5$  and  $\phi_6$  used to bosonize the fermions. The Baker–Campbell–Hausdorff formula implies

$$e^{i\alpha q_i} e^{i\beta p_j} = e^{-i\alpha\beta\delta_{ij}} e^{i\beta p_j} e^{i\alpha q_i}, \quad (\text{A.36})$$

which one can use to confirm that this operator has the correct action on fermions.

From this perspective, one can see that any operator that is ‘pure exponential’ in  $\phi_5$  and  $\phi_6$  transforms as in Equation (A.27). If one considers any of the chiral primaries of the  $\mathcal{N} = 4$  orbifold theory, one finds that all of the chiral primaries are ‘pure exponential’ and therefore transform using Equation (A.27).

One finds that the the currents transform under spectral flow as follows

$$\begin{aligned} J^3(z) &\mapsto J^3(z) - \frac{c\alpha}{12z} \\ J^\pm(z) &\mapsto z^{\mp\alpha} J^\pm(z) \\ G^{\pm A}(z) &\mapsto z^{\mp\frac{\alpha}{2}} G^{\pm A}(z) \\ T(z) &\mapsto T(z) - \frac{\alpha}{z} J^3(z) + \frac{c\alpha^2}{24z^2}, \end{aligned} \quad (\text{A.37})$$

which gives rise to the transformation of the modes,

$$\begin{aligned} J_m^3 &\mapsto J_m^3 - \frac{c\alpha}{12} \delta_{m,0} \\ J_m^\pm &\mapsto J_{m\mp\alpha}^\pm \\ G_m^{\pm A} &\mapsto G_{m\mp\frac{\alpha}{2}}^{\pm A} \\ L_m &\mapsto L_m - \alpha J_m^3 + \frac{c\alpha^2}{24} \delta_{m,0}. \end{aligned} \quad (\text{A.38})$$

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<sup>6</sup>There are implicit cocycles on the exponentials, which make unrelated fermions anticommute. Thus, the order of exponentials in expressions matters.

Spectral flow also acts on states, changing the weight and charge by

$$h \mapsto h' = h + \alpha m + \frac{c\alpha^2}{24} \quad (\text{A.39a})$$

$$m \mapsto m' = m + \frac{c\alpha}{12}, \quad (\text{A.39b})$$

which can be read off from  $L_0$  and  $J_0^3$ . Frequently, one can deduce the spectral-flowed state from the weight and charge.

Note that spectral flow by  $\alpha_1$  units followed by spectral flow by  $\alpha_2$  units is equivalent to spectral flow by  $\alpha_1 + \alpha_2$  units. Therefore, spectral flow forms an abelian group, and

$$\mathcal{U}_\alpha^{-1} = \mathcal{U}_{-\alpha}. \quad (\text{A.40})$$

### A.10 Ramond sector

The CFT in the complex plane naturally has periodic fermions, which corresponds to the NS sector. One can, however, spectral flow by an odd number of units to the Ramond sector. If one starts with the NS vacuum and then spectral flows with  $\alpha = -1$ , then the state in the R sector has

$$h = \frac{1}{4} \quad m = \frac{1}{2}. \quad (\text{A.41})$$

From the weight we see that this must be the R ground state. Let us call this state

$$|\emptyset\rangle_R^+. \quad (\text{A.42})$$

Since, we are now in the Ramond ground state we have fermion zero modes, and therefore may act with  $J_0^-$ , which gives us the state

$$|\emptyset\rangle_R^- = J_0^- |\emptyset\rangle_R^+ = -\frac{1}{2}\epsilon_{\dot{A}\dot{B}}\psi_0^{-\dot{A}}\psi_0^{-\dot{B}} |\emptyset\rangle_R^+ = \psi_0^{-\dot{2}}\psi_0^{-\dot{1}} |\emptyset\rangle_R^+. \quad (\text{A.43})$$

The normalization is fixed by the commutation relations of  $J_0^a$ . Since  $J_0^-$  has zero weight, one can be sure that this state is also a member of the R vacuum. Acting twice with  $J_0^-$  annihilates the state, from which one concludes that these states form a doublet of  $SU(2)_L$ ,

$$|\emptyset\rangle_R^\alpha, \quad (\text{A.44})$$

and one also can determine that

$$|\emptyset\rangle_R^+ = J_0^+ |\emptyset\rangle_R^- = \frac{1}{2}\epsilon_{\dot{A}\dot{B}}\psi_0^{+\dot{A}}\psi_0^{+\dot{B}} |\emptyset\rangle_R^- = \psi_0^{+\dot{1}}\psi_0^{+\dot{2}} |\emptyset\rangle_R^-. \quad (\text{A.45})$$

What happens if we act on these states not with a pair of fermion zero modes in the current  $J$ , but with a single fermion zero mode directly? Since one cannot raise the charge of the state  $|\emptyset\rangle_R^+$  or lower the charge of the state  $|\emptyset\rangle_R^-$ , one must have

$$\psi_0^{+\dot{A}} |\emptyset\rangle_R^+ = 0 \quad \psi_0^{-\dot{A}} |\emptyset\rangle_R^- = 0. \quad (\text{A.46})$$

This can also be seen from Equations (A.43) and (A.45). However, one ought to be able to contract the fermion zero mode index with the R vacuum doublet index to form

$$|\emptyset\rangle_R^{\dot{A}} = \frac{1}{\sqrt{2}} \epsilon_{\alpha\beta} \psi_0^{\alpha\dot{A}} |\emptyset\rangle_R^\beta, \quad (\text{A.47})$$

where the normalization is determined from the fermion mode anticommutation relations. Since there are four fermion zero modes (in the left sector), we expect four Ramond vacuum. We see that those vacua form a doublet of  $SU(2)_L$  and a doublet of  $SU(2)_2$ .

Of course, the same story holds on the right sector of the theory as well, which gives a total of 16 Ramond vacua:

$$|\emptyset\rangle_R^{\alpha\dot{\alpha}} \quad |\emptyset\rangle_R^{\dot{A}\dot{\alpha}} \quad |\emptyset\rangle_R^{\alpha\dot{A}} \quad |\emptyset\rangle_R^{\dot{A}\dot{B}}. \quad (\text{A.48})$$

Note that we must be very careful to always write the index corresponding to the left zero modes first and the index corresponding to the right zero modes second.

What are the images of the Ramond vacua in the NS sector? From the action of spectral flow on the charge and weight of a state, one can conclude that the Ramond vacua are one unit of spectral flow from chiral primary ( $h = m$ ) states in the NS sector; or equivalently, negative one units of spectral flow from anti-chiral primary ( $h = -m$ ) states. Therefore, there is a one-to-one correspondence between the R vacua and the NS chiral primary states. There are four left chiral primary states in the NS sector,

$$|\emptyset\rangle_{NS} \quad \psi_{-\frac{1}{2}}^{+\dot{A}} |\emptyset\rangle_{NS} \quad \epsilon_{\dot{A}\dot{B}} \psi_{-\frac{1}{2}}^{+\dot{A}} \psi_{-\frac{1}{2}}^{+\dot{B}} |\emptyset\rangle_{NS}, \quad (\text{A.49})$$

and there are also four states in the right sector. Thus a total of 16 chiral primary states in the NS sector that get mapped onto the 16 Ramond vacua via spectral flow. These are all of the chiral primary states for a single strand of the CFT. In the twisted sector, there are more chiral primary states which correspond to Ramond ‘vacua’ in the twisted sector.

### A.11 Chiral primaries

Consider the supercurrents in the anomaly free subalgebra. In particular we have

$$\{G_{+\frac{1}{2}}^{-A}, G_{-\frac{1}{2}}^{+B}\} = 2\epsilon^{AB}(J_0^3 - L_0). \quad (\text{A.50})$$

Since

$$(G_{-\frac{1}{2}}^{+B})^\dagger = -\epsilon_{BC} G_{+\frac{1}{2}}^{-C}, \quad (\text{A.51})$$

one finds that for a normalized state  $|\psi\rangle$  of weight  $h$  and charge  $m$

$$\begin{aligned} \langle\psi| (G_{-\frac{1}{2}}^{+B})^\dagger G_{-\frac{1}{2}}^{+B} |\psi\rangle &= -\epsilon_{BC} \langle\psi| G_{+\frac{1}{2}}^{-C} G_{-\frac{1}{2}}^{+B} |\psi\rangle \\ &= \epsilon_{BC} \langle\psi| G_{-\frac{1}{2}}^{+B} G_{+\frac{1}{2}}^{-C} |\psi\rangle - 2\epsilon_{BC}\epsilon^{CB} \langle\psi| (J_0^3 - L_0) |\psi\rangle \\ &= -\langle\psi| (G_{+\frac{1}{2}}^{-C})^\dagger G_{+\frac{1}{2}}^{-C} |\psi\rangle - 2(m - h) \\ \implies \langle\psi| (G_{-\frac{1}{2}}^{+B})^\dagger G_{-\frac{1}{2}}^{+B} |\psi\rangle &= h - m. \end{aligned} \quad (\text{A.52})$$



This and a similar calculation from the anticommutation relation

$$\{G_{+\frac{1}{2}}^{+A}, G_{-\frac{1}{2}}^{-B}\} = 2\epsilon^{AB}(J_0^3 + L_0), \quad (\text{A.53})$$

imply that for our theory to be unitary, we need all physical states to have weight greater than (the absolute value of) charge:

$$h \geq |m|. \quad (\text{A.54})$$

A *chiral* state is a state that is annihilated by  $G_{-\frac{1}{2}}^{+A}$  for  $A = 1, 2$ . A *primary* state is killed by all the positive modes of the theory. Thus, a state or operator that is both chiral and primary must saturate the above bound, having  $h = m$ . In fact, the converse is also true: any state or operator that saturates the bound is a chiral primary.

Since an  $h = m$  state  $|\chi\rangle$  saturates the bound, its weight cannot be lowered without lowering the charge, and thus

$$L_m |\chi\rangle = J_m^3 |\chi\rangle = 0 \quad m > 0. \quad (\text{A.55})$$

From the bound, one can immediately conclude that

$$G_{+\frac{1}{2}}^{+A} |\chi\rangle = 0. \quad (\text{A.56})$$

On the other hand, it is also possible to show

$$G_{+\frac{1}{2}}^{-A} |\chi\rangle = 0 \quad G_{-\frac{1}{2}}^{+A} |\chi\rangle = 0. \quad (\text{A.57})$$

To demonstrate this, one must use the algebra to show that the sum of the norms of the above two states must vanish. Then, the above follows from the positive-definiteness of the norm. Therefore, the states saturating the bound are annihilated by all of the  $G$ 's in the anomaly-free subalgebra except for  $G_{-1/2}^{-A}$ . Since all of the positive modes kill the state, the state must also be primary. That is,  $h = m$  implies chiral primary.

The importance of the chiral primary operators is that they correspond to the top member<sup>7</sup> of the superconformal multiplets *in the NS sector*. Therefore, it suffices to find the chiral primary operators to catalog the representations of the superconformal algebra.

## B. Cartesian to spherical Clebsch–Gordan coefficients

In this section, we outline our conventions for relating irreducible spherical tensors to ordinary cartesian tensors in flat space. This fixes the factors in going from Equation (2.52) to Equation (2.63) for the D1D5 case, and explains how we define the correctly normalized differential operator, so that Equation (2.39) is satisfied.

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<sup>7</sup>Frequently, this is called the ‘highest weight state’, even though it actually has the lowest conformal mass dimension of the multiplet.

Our goal is to show how to construct the coefficients,  $Y_{l,m_\psi,m_\phi}^{j_1 \dots j_l}$ , that define the differential operator in Equation (2.38) such that it satisfies Equation (2.39),

$$Y_{l,m_\psi,m_\phi}^{k_1 k_2 \dots k_l} \partial_{k_1} \partial_{k_2} \dots \partial_{k_l} [r^{l'} Y_{l',m'_\psi,m'_\phi}(\Omega_3)] = \delta_{ll'} \delta_{m_\psi m'_\psi} \delta_{m_\phi m'_\phi}.$$

We take the spherical harmonics as a starting point. The cartesian coordinates for the non-compact space are related to the angular coordinates via

$$\begin{aligned} x^1 &= r \cos \theta \cos \psi \\ x^2 &= r \cos \theta \sin \psi \\ x^3 &= r \sin \theta \cos \phi \\ x^4 &= r \sin \theta \sin \phi, \end{aligned} \tag{B.1}$$

where the  $(\theta, \psi, \phi)$  are restricted to

$$\theta \in [0, \frac{\pi}{2}) \quad \psi, \phi \in [0, 2\pi). \tag{B.2}$$

Spherical harmonics can be written as a homogeneous polynomial of the cartesian unit vector components of the form

$$Y_{l,m_\psi,m_\phi}(\Omega_3) = \frac{1}{r^l} \mathcal{Y}_{j_1 \dots j_l}^{l,m_\psi,m_\phi} x^{j_1} \dots x^{j_l} = \mathcal{Y}_{j_1 \dots j_l}^{l,m_\psi,m_\phi} n^{j_1} \dots n^{j_l}. \tag{B.3}$$

The  $\mathcal{Y}$  must be pairwise symmetric and traceless. One can compute these tensors by the usual methods of breaking up a tensor into irreducible components, or by inspection of the explicit form of the spherical harmonics in angular components.

Given the spherical harmonic normalization

$$\int Y_{l,m_\psi,m_\phi}^* Y_{l',m'_\psi,m'_\phi} d\Omega = \delta_{ll'} \delta_{m_\psi m'_\psi} \delta_{m_\phi m'_\phi}, \tag{B.4}$$

one can determine an orthogonality relation for the  $\mathcal{Y}$ 's:

$$(\mathcal{Y}_{j_1 \dots j_l}^{l,m_\psi,m_\phi})^* \mathcal{Y}_{k_1 \dots k_{l'}}^{l',m'_\psi,m'_\phi} \int (n^{j_1} \dots n^{j_l})(n^{k_1} \dots n^{k_{l'}}) d\Omega = \delta_{ll'} \delta_{m_\psi m'_\psi} \delta_{m_\phi m'_\phi}. \tag{B.5}$$

The integral over  $l + l'$  unit vectors defines a natural inner product on the Clebsch–Gordan coefficients  $\mathcal{Y}$ .

We label the integral

$$I^{j_1 \dots j_l k_1 \dots k_{l'}}, \tag{B.6}$$

and note that  $I$  must be symmetric in all of its indices. Furthermore, from the symmetry of the integral one must conclude that  $I$  vanishes unless every index appears an even number of times. For instance,

$$I^j = \int n^j d\Omega = 0. \tag{B.7}$$

As a corollary,  $I$  vanishes unless it has an even number of indices. Having picked off the easiest properties, let's without further comment give the general form. Let  $a_i$  be the total number of times the index  $i$  appears in the collection of indices on  $I$ , then

$$\begin{aligned} I^{[a_1, a_2, a_3, a_4]} &= \int (n^1)^{a_1} (n^2)^{a_2} (n^3)^{a_3} (n_4)^{a_4} d\Omega_3 \\ &= \left[ \int_0^{\frac{\pi}{2}} \cos^{a_1+a_2+1} \theta \sin^{a_3+a_4+1} \theta d\theta \right] \left[ \int_0^{2\pi} \cos^{a_1} \psi \sin^{a_2} \psi d\psi \right] \left[ \int_0^{2\pi} \cos^{a_3} \phi \sin^{a_4} \phi d\phi \right]. \end{aligned} \quad (\text{B.8})$$

We recognize the above definite integrals as different representations of the beta function:

$$\int_0^{\frac{\pi}{2}} \cos^\alpha \theta \sin^\beta \theta d\theta = \frac{1}{2} B\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right) \quad (\text{B.9a})$$

$$\int_0^{2\pi} \cos^\alpha \theta \sin^\beta \theta d\theta = \frac{1}{2} [1 + (-1)^\alpha + (-1)^{\alpha+\beta} + (-1)^\beta] B\left(\frac{\alpha+1}{2}, \frac{\beta+1}{2}\right), \quad (\text{B.9b})$$

where the second equation follows from the first. Therefore, one sees that *provided all the  $a_i$  are even*

$$\begin{aligned} I^{[a_1, a_2, a_3, a_4]} &= 2B\left(\frac{a_1+a_2+2}{2}, \frac{a_3+a_4+2}{2}\right) B\left(\frac{a_1+1}{2}, \frac{a_2+1}{2}\right) B\left(\frac{a_3+1}{2}, \frac{a_4+1}{2}\right) \\ &= 2\pi^2 \left( \frac{1}{\pi^2} \frac{\Gamma\left(\frac{a_1+1}{2}\right) \Gamma\left(\frac{a_2+1}{2}\right) \Gamma\left(\frac{a_3+1}{2}\right) \Gamma\left(\frac{a_4+1}{2}\right)}{\Gamma\left(\frac{a_1+a_2+a_3+a_4+4}{2}\right)} \right) \\ &= \frac{2\pi^2 (a_1-1)! (a_2-1)! (a_3-1)! (a_4-1)!}{2^{a_1+a_2+a_3+a_4-4} \left(\frac{a_1}{2}-1\right)! \left(\frac{a_2}{2}-1\right)! \left(\frac{a_3}{2}-1\right)! \left(\frac{a_4}{2}-1\right)! \left(\frac{a_1+a_2+a_3+a_4}{2}+1\right)!} \quad (\text{B.10}) \end{aligned}$$

Note that in the last line one must use the limit

$$\lim_{x \rightarrow 0} \frac{(x-1)!}{\left(\frac{x}{2}-1\right)!} = \frac{1}{2}, \quad (\text{B.11})$$

in the event that some of the  $a_i$  are zero.

The orthogonality condition from Equation (B.5) is

$$I^{j_1 \dots j_l k_1 \dots k_{l'}} (\mathcal{Y}_{j_1 \dots j_l}^{l, m_\psi, m_\phi})^* \mathcal{Y}_{k_1 \dots k_{l'}}^{l', m'_\psi, m'_\phi} = \delta_{ll'} \delta_{m_\psi m'_\psi} \delta_{m_\phi m'_\phi}, \quad (\text{B.12})$$

which motivates the choice of

$$Y_{l, m_\psi, m_\phi}^{j_1 \dots j_l} \propto I^{j_1 \dots j_l k_1 \dots k_l} (\mathcal{Y}_{k_1 \dots k_l}^{l, m_\psi, m_\phi})^*. \quad (\text{B.13})$$

We can think of the  $2l$ -index  $I$  as defining an inner product on the space of symmetric traceless  $l$ -index tensors, spanned by the  $\mathcal{Y}_{j_1 \dots j_l}^{l, m_\psi, m_\phi}$ . Then, we can think of  $Y_{l, m_\psi, m_\phi}$  as (proportional to) the dual of  $\mathcal{Y}_{j_1 \dots j_l}^{l, m_\psi, m_\phi}$ .

Since the  $l$  derivatives are symmetrized and the spherical harmonic's cartesian form is also symmetrized, we get a factor of  $l!$ . One finds that

$$I^{j_1 \dots j_l k_1 \dots k_l} (\mathcal{Y}_{k_1 \dots k_l}^{l, m_\psi, m_\phi})^* \partial_{j_1} \dots \partial_{j_l} r^l Y_{l, m_\psi, m_\phi}(\theta, \psi, \phi) \Big|_{r \rightarrow 0} = l! \quad (\text{B.14})$$

and therefore we define

$$Y_{l, m_\psi, m_\phi}^{j_1 \dots j_l} = \frac{1}{l!} I^{j_1 \dots j_l k_1 \dots k_l} (\mathcal{Y}_{k_1 \dots k_l}^{l, m_\psi, m_\phi})^* . \quad (\text{B.15})$$

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